# Solution to sheet 7

**Discussion topic:** Relativistic fluid dynamics: what are the fundamental equations? Which further ingredients are needed to model a specific system? Which assumption(s) underlie the description?

The fundamental equations of relativistic fluid dynamics are the conservation of the currents and the energy-momentum tensor conservation equation.

In order to model a specific system one needs at least the equation of state which is a relation between the pressure, the energy density and the chemical potentials in the most general case. Other ingredient might be the transport coefficients and an initial state.

The assumptions made are mainly that the system can be described as a fluid in the sense that one can define cells in which local equilibrium exists and therefore the temperature can be defined and all thermodynamic properties. This cells must be small enough such that gradients can be neglected. Other assumptions might be using ideal hydrodynamics.

## 14. Local thermodynamics

In your Statistical Physics lectures, thermodynamics and in particular the fundamental thermodynamic relations were formulated for extensive quantities, as e.g.

$$U = TS - \mathcal{P}\mathcal{V} + \sum_a \mu_a N_a \quad , \quad \mathrm{d}U = T\,\mathrm{d}S - \mathcal{P}\mathrm{d}\mathcal{V} + \sum_a \mu_a\,\mathrm{d}N_a$$

for a system with several conserved quantum numbers. The purpose of the present exercise is to derive the equivalent relations between the densities of the extensive thermodynamic parameters: energy density[1] $\epsilon$, entropy density $s$, number densities $n_a$.[2]

**i.** The relation for the internal energy $U$ translate at once in a relation for $\epsilon$. Using the latter and the relation for $\mathrm{d}U$, show that the differentials of energy density and pressure obey

$$\mathrm{d}\epsilon = T\,\mathrm{d}s + \sum_a \mu_a\,\mathrm{d}n_a \quad , \quad \mathrm{d}\mathcal{P} = s\mathrm{d}T + \sum_a n_a\,\mathrm{d}\mu_a \tag{1}$$

respectively. The second of these relations is the local version of the *Gibbs–Duhem relation*.

**Solution:** The energy density and the internal energy are related by the work done. So basically if we subtract the work (P dV) from the second expression we obtain:

$$dE = T\,\mathrm{d}S + \sum_a \mu_a\,\mathrm{d}N_a,$$

which, if we divide by the volume becomes

$$\mathrm{d}\epsilon = T\mathrm{d}s + \sum_a \mu_a\,\mathrm{d}n_a \quad . \tag{2}$$

For the pressure we could differentiate the first equation to find

$$dU = SdT + TdS - \mathcal{P}d\mathcal{V} - \mathcal{V}d\mathcal{P} + \sum_a (N_a d\mu_a + \mu_a dN_a) \tag{3}$$

[1] The equations take the same for in the non-relativistic and the relativistic cases.

[2] Note that these are the volume densities, not the specific densities used in some textbooks.

and by comparing it with the second equation and later dividing by the volume we find

$$\mathrm{d}\mathcal{P} = s\mathrm{d}T + \sum_a n_a \,\mathrm{d}\mu_a \tag{4}$$

Notice that in the absence of conserved quantum numbers we have $d\epsilon + d\mathcal{P} = Tds + sdT \to e + p = Ts$

**ii.** Focusing on the case with no conserved quantum number, deduce from the previous relations an expression for the derivative

$$c_s^2 \equiv \frac{\partial \mathcal{P}}{\partial \epsilon} \tag{5}$$

in terms of the entropy density $s$ and the temperature $T$.

Assuming that $c_s^2$ is a constant, in which case Eq. (5) is equivalent to the simple "equation of state" $\mathcal{P} = c_s^2 \epsilon$, show the scaling behaviors

$$\epsilon \propto T^{1+1/c_s^2} \quad , \quad s \propto T^{1/c_s^2}. \tag{6}$$

What do you find in the case $\epsilon = 3\mathcal{P}$? Since this is the equation of state for an ideal gas of ultra-relativistic particles, you may refresh your knowledge on the photon gas from your Statistical Physics lectures.

**Solution:**

If there are no conserved quantum numbers the energy density and pressure become simply

$$\mathrm{d}\epsilon = T\,\mathrm{d}s \quad , \quad \mathrm{d}\mathcal{P} = s\mathrm{d}T \tag{7}$$

The speed of sound is precisely defined as how the pressure changes for a given change of energy density, therefore we can write

$$c_s^2 = \frac{d\mathcal{P}}{d\epsilon} = \frac{s}{T}\frac{dT}{ds}$$

If $c_s^2 \epsilon = \mathcal{P}$ the entropy becomes

$$\frac{ds}{s} = \frac{1}{c_s^2}\frac{dT}{T} \to s \propto T^{1/c_s^2}.$$

The energy density (and similarly the pressure) can be obtained as

$$d\mathcal{P} = sdT \propto T^{1/c_s^2} dT \to c_s^2 \epsilon = \mathcal{P} \propto T^{1+1/c_s^2}.$$

In the case that $c_s^2 = 1/3$ the solution becomes

$$s \propto T^3 \qquad \epsilon \propto T^4 \propto \mathcal{P}$$

This are the typical results used for many applications as for modeling heavy-ion collisions for example (at least up to a first approximation). If you want to do better then you use an equation of state provided by lattice QCD calculations which is far more complicated but in principle it is also far more realistic than an ideal gas.

## 15. Relativistic fluid dynamics

The dynamics of a perfect relativistic fluid without conserved charge is entirely governed by the equation $\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu} = \epsilon u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu}$, with $u^\mu$ the 4-velocity and $\Delta^{\mu\nu} \equiv u^\mu u^\nu + g^{\mu\nu}$, where the mostly plus convention is used for the metric tensor (so that $u_\mu u^\mu = -1$).

Throughout the exercise, $c = 1$ and the x-dependence of fields is not written.

The fundamental equation of motion implies *automatically* entropy conservation, $\partial_\mu(su^\mu) = 0$. Using thermodynamic relations from question **i.** of the previous exercise, show that in a perfect relativistic fluid the temperature evolves according to the equation

$$u^\mu \partial_\mu (Tu^\nu) = -\partial^\nu T. \tag{8}$$

Deduce from the latter and from a result from exercise **14.ii.** the equation

$$u^\mu \partial_\mu u^\nu = -c_s^2 \frac{\nabla^\nu s}{s}, \tag{9}$$

where $\nabla^\nu \equiv \Delta^{\nu\mu}\partial_\mu$. How can you interpret this equation?

**Solution:**

$$T^{\mu\nu} = (e+P)u^\mu u^\nu + Pg^{\mu\nu} = Tsu^\mu u^\nu + Pg^{\mu\nu} = (su^\mu)(Tu^\nu) + Pg^{\mu\nu}$$

Using the conservation equations of the energy-momentum tensor we find

$$\partial_\mu T^{\mu\nu} = su^\mu \partial_\mu (Tu^\nu) + Tu^\nu \partial_\mu (su^\mu) + \partial^\nu \mathcal{P} = su^\mu \partial_\mu (Tu^\nu) + \partial^\nu \mathcal{P} = 0$$

From the thermodynamics relations we have deduced before we have $d\mathcal{P} = sdT \to \partial^\nu \mathcal{P} = s\partial^\nu T$.

Therefore,

$$su^\mu \partial_\mu (Tu^\nu) = -s\partial^\nu T \to u^\mu \partial_\mu (Tu^\nu) = -\partial^\nu T$$

We will now rewrite the above equation making use of $c_s^2 = \frac{s}{T}\frac{dT}{ds}$.

First we expand the left hand side ,

$$u^\mu u^\nu \partial_\mu T + Tu^\mu \partial_\mu u^\nu = -\partial^\nu T$$

Next we rearrange some terms as

$$Tu^\mu \partial_\mu u^\nu = -\partial^\nu T - u^\mu u^\nu \partial_\mu T = -(g^{\mu\nu}\partial_\mu + u^\mu u^\nu \partial_\mu)T = -\nabla^\nu T$$

$$u^\mu \partial_\mu u^\nu = -\frac{\nabla^\nu T}{T}$$

Lastly we introduce the entropy to find

$$u^\mu \partial_\mu u^\nu = -c_s^2 \frac{\nabla^\nu s}{s}$$