Solution to sheet 6

12. A toy model of multiparticle correlations and cumulants

The purpose of this exercise is to illustrate the meaning of the cumulants of multiparticle azimuthal averages on a simple example of particle emission with two- or three-particle correlations. Throughout the exercise, $\langle\cdots\rangle$ denotes an average over particles and events (in two successive steps).

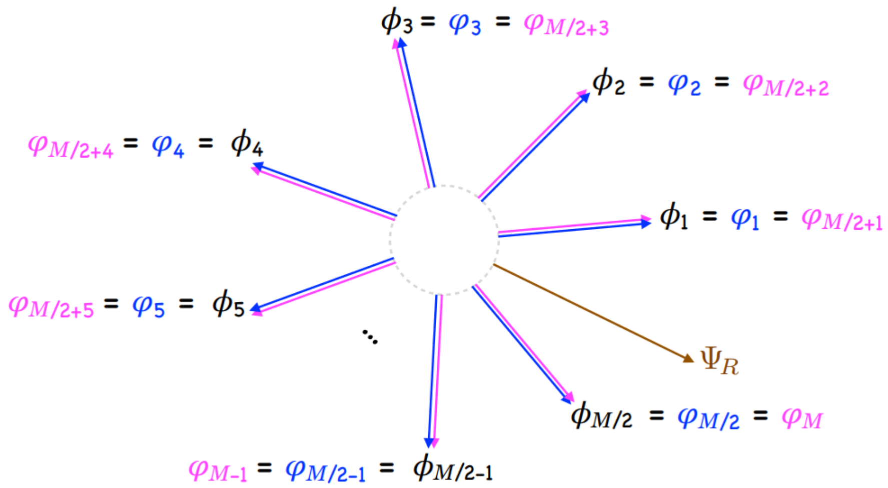
i. Two-particle correlations

Consider first a set of events, such that in each event exactly M particles are emitted as follows: an angle Ψ_R and $M/2$ other angles ϕ_j are chosen randomly with an isotropic distribution. Then for each $j = 1, 2, \ldots, M/2$, two particles (labeled j and $M/2 + j$) are emitted with the same azimuth $\varphi_j = \varphi_{M/2+j} = \phi_j$.

a) What is the value of the anisotropic flow coefficients $v_n = \left\langle e^{in(\varphi_j - \Psi_R)}\right\rangle$ (for any $n \in \mathbb{N}^*$)? Same question for the single- and two-particle averages $\left\langle e^{in\varphi_j}\right\rangle$ and $\left\langle e^{in(\varphi_j+\varphi_k)}\right\rangle$ with $j \neq k$.

Hint: No explicit calculation is needed!

Solution:



We want to compute $v_n = \left\langle e^{in(\varphi_j - \Psi_R)}\right\rangle$, however, since the angles of the pairs are isotropically distributed and the reaction plane angles are also isotropic this will give zero and equivalently for the single and two particles averages.

Just as an example we can perform one of them explicitly

$$v_n = \left\langle e^{in(\varphi_j - \Psi_R)}\right\rangle_{particles,events} \propto \sum_{events}\sum_{particles} e^{in\varphi_j} e^{-in\Psi_R} = 0.$$

The above equation is not fully true. If the number of particles is finite one might get that the average over particles is different than zero for a given event, but when averaging over events it will give zero.

b) Compute for any $n \in \mathbb{N}^*$ the two-particle average $\left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j - \varphi_k)} \right\rangle$ with $j \neq k$. (*Hint*: You may want to distinguish the two cases $|j-k| \neq M/2$ and $|j-k| = M/2$ and count how often each of these two possibilities occur in a given event).

Define the two-particle azimuthal cumulant as $c_n\{2\} \equiv \left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j - \varphi_k)} \right\rangle - \left\langle \mathrm{e}^{\mathrm{i}n\varphi_j} \right\rangle \left\langle \mathrm{e}^{-\mathrm{i}n\varphi_k} \right\rangle$ and a flow estimate $v_n\{2\}$ by $v_n\{2\}^2 \equiv c_n\{2\}$. What do you find for $v_n\{2\}$?

Solution: Now we compute $\left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j - \varphi_k)} \right\rangle_{j \neq k}$. For the case $|j-k| \neq M/2$ this will give zero since one again the two angles are isotropically distributed. Therefore, just the case $|j-k| = M/2$ has to be considered.

Hence,

$$\left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j - \varphi_k)} \right\rangle_{|j-k|=M/2} = < e^0 > = \frac{1}{M(M-1)} M = \frac{1}{M-1} \approx \frac{1}{M}$$

where the number of pairs is $N_{pairs} = M(M-1)$ and for each particle there is another particle that is correlated with this one. (You might have in mind $N_{pairs} = \frac{N(N-1)}{2}$ from statistical physics but the factor 2 is not there because it corresponds to a system of indistinguishable particles).

The two particle azimuthal cumulant $c_n\{2\} \equiv \left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j - \varphi_k)} \right\rangle - \left\langle \mathrm{e}^{\mathrm{i}n\varphi_j} \right\rangle \left\langle \mathrm{e}^{-\mathrm{i}n\varphi_k} \right\rangle$ is easy to compute since the we have already computed the first term and the other two are zero as we have showed before.

An estimation to the flow is $v_n\{2\}^2 \equiv c_n\{2\}$ which leads to

$$v_n\{2\} = \frac{1}{\sqrt{M-1}}.$$

This means that there will be anisotropic flow coefficients even if there should not be any since particles are generated isotropically. We have seen before (part a) that $v_n = 0$ but with the cumulants method all $v_n\{2\}$ get a non-zero contribution usually referred as non-flow effects. Notice that such contribution vanishes in the limit $M \to \infty$.

c) Compute for any $n \in \mathbb{N}^*$ the four-particle average $\left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j + \varphi_k - \varphi_l + \varphi_p)} \right\rangle$ where the four particles are all different. Define the four-particle azimuthal cumulant as

$$c_n\{4\} \equiv \left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j + \varphi_k - \varphi_l - \varphi_p)} \right\rangle - \left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j - \varphi_l)} \right\rangle \left\langle \mathrm{e}^{-\mathrm{i}n(\varphi_k - \varphi_p)} \right\rangle - \left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j - \varphi_p)} \right\rangle \left\langle \mathrm{e}^{-\mathrm{i}n(\varphi_k - \varphi_l)} \right\rangle$$

and a flow estimate $v_n\{4\}$ by $v_n\{4\}^4 \equiv -c_n\{4\}$. What do you find for $v_n\{4\}$? Can you justify the definition of $c_n\{4\}$? (why are there no extra terms?)

Solution:

The four-particle azimuthal cumulant has a lengthy form but we know the last terms since we have just computed them before. Using that information we can write

$$c_n\{4\} = \left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j + \varphi_k - \varphi_l - \varphi_p)} \right\rangle - \frac{2}{M^2}$$

We will compute $\left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j + \varphi_k - \varphi_l - \varphi_p)} \right\rangle$. Again many cases will not give any contribution. We know that only the cases where $|j+k-l-p| = M$ and at the same time $|j-l| = M/2$ or $|j-p| = M/2$ will contribute. That means the four particles belong to two pairs where each pair contains one positive and one negative sign. First, the number of quadruplets we can create are $N_{quadrup} = M(M-1)(M-2)(M-3) \approx M^4$.

We are left with

$$\left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j+\varphi_k-\varphi_l-\varphi_p)} \right\rangle = \frac{1}{M^4} \sum_{j,k,l,p;j\neq k\neq l\neq p} \mathrm{e}^{\mathrm{i}n(\varphi_j+\varphi_k-\varphi_l-\varphi_p)} = \frac{1}{M^4} M(M-2)\cdot 2\cdot 1 \approx \frac{2}{M^2}$$

where the sum gives M particles which for each particle there are (M-2) other particles that have a different angle and then for each of these two particles there are two particles that have and angle equal to the one of the first two and one particle is left that closes the two pairs. (Try to do it by yourself and argument it).

If you trust the above calculations the four-particle azimuthal cumulant and the associated flow read as

$$c_n\{4\} = \frac{2}{M^2} - \frac{2}{M^2} = 0; \qquad v_n\{4\} = 0.$$

This is actually not a surprise since we are trying to compute the fourth particle cumulant but out particle distribtuion is a two particle distribution (sort to speak) such that there are no fourth particle correlations but only two particle correlations. This also explains why $c_n\{4\}$ is defined as it is defined and it is not just defined as the first term. It is because otherwise it would NOT be zero even if there were no four-particle correlations in the system.

ii. Three-particle correlations

(optional: estimate first if you will be able to solve the following in a quarter of hour or less)

Assume now that each event consists of $M/3$ isotropically distributed triplets of particles with the same azimuth $\varphi_j = \varphi_{M/3+j} = \varphi_{2M/3+j} = \phi_j$ — generalizing the setup of question **i.**.

Repeat the calculations of questions **i.b)** and **i.c)** for the new setup.

Solution:

Part **c)** is easy to answer, since the system has three-particles correlation, the fourth-particle cumulant would be zero as discussed before. Part **b)** is similar to what we have done before and I will let you explicitly compute it. The solution is

$$c_n\{2\} = \left\langle \mathrm{e}^{\mathrm{i}n(\varphi_j-\varphi_k)} \right\rangle = \frac{2}{M-1}; \qquad v_n\{2\} = \sqrt{\frac{2}{M-1}}.$$

Where the cumulant is now twice what it was in section b). Can you explain why?

(Answer: Because for each particle there are two particles that are correlated with that one meanwhile in the other case for each particle there was one that was correlated with it. This explains the factor two.)

13. Momentum conservation

Consider M particles such that the sum of their (transverse) momenta vanishes:

$$\mathbf{p}_1 + \mathbf{p}_2 + \cdots \mathbf{p}_M = \mathbf{0}. \tag{1}$$

For simplicity, we assume that all momenta have the same modulus: $|\mathbf{p}_1| = \cdots = |\mathbf{p}_M|$.

Let φ_j denote the azimuthal angle of $\mathbf{p}_j$ with respect to some fixed direction. Assuming that the azimuths are isotropically distributed — which somehow implies $M \gg 1$ to make sense — leads automatically to $\langle \cos\varphi_j \rangle = 0$. Nevertheless, and possibly contrary to your intuition, $\langle \cos(\varphi_j - \varphi_k) \rangle$ with $j \neq k$ is non-zero. More precisely, show that Eq. (1) implies

$$\langle \cos(\varphi_j - \varphi_k) \rangle \underset{M\gg 1}{\simeq} -\frac{1}{M} \tag{2}$$

and give a one-sentence physical interpretation of that result.

Hint: You may want to square Eq. (1).

Solution:

As suggested we start by squaring Eq.(1) as follows

$$(\mathbf{p}_1+\mathbf{p}_2+\cdots\mathbf{p}_M)^2 = (\mathbf{p}_1^2+...+\mathbf{p}_M^2)+2(\mathbf{p}_1\cdot\mathbf{p}_2+...+\mathbf{p}_{M-1}\cdot\mathbf{p}_M) = M\mathbf{p}_M^2+2(\mathbf{p}_1\cdot\mathbf{p}_2+...+\mathbf{p}_{M-1}\cdot\mathbf{p}_M) = 0 \tag{3}$$

We want to compute $\langle\cos(\varphi_i-\varphi_j)\rangle$. Since we have a discrete distribution this reads as

$$\langle\cos(\varphi_i-\varphi_j)\rangle = \left\langle \frac{p_i^x p_j^x + p_i^y p_j^y}{|\mathbf{p}_i||\mathbf{p}_j|} \right\rangle_{i\neq j} = \frac{1}{N_{pairs}p_M^2}\sum_{i\neq j} p_i^x p_j^x + p_i^y p_j^y = \frac{2}{M(M-1)p_M^2}(\mathbf{p}_1\cdot\mathbf{p}_2+...+\mathbf{p}_{M-1}\cdot\mathbf{p}_M)$$

$$= -\frac{1}{M-1} \approx -\frac{1}{M}$$

Momentum conservation makes particles be correlated. One can think about it as: All of the M particles can have any random momentum you want except the last one which has to fulfill the momentum conservation equation. Therefore one out of M particles is not "free" hence $\frac{1}{M}$. As usual, if M is large this correlations disappears.