Solutions to sheet 5

10. Ellipticity and quadrangularity

In exercise **7.i.** you have computed the function $\mathrm{d}^2T_{AA}(\mathbf{b})/\mathrm{d}^2\mathbf{s} \equiv T_A(\mathbf{s}-\mathbf{b}/2)T_A(\mathbf{s}+\mathbf{b}/2)$ as a function of the impact parameter (modulus) b for the collision between two lead nuclei, with $T_A(\mathbf{s})$ the nuclear thickness function. Using $\mathrm{d}^2T_{AA}(\mathbf{b})/\mathrm{d}^2\mathbf{s}$ as a weight, compute and plot as a function of b the "ellipticity" and "quadrangularity"

$$\epsilon_2^{(\mathbf{r})} \equiv -\frac{\langle r^2\cos(2\theta)\rangle}{\langle r^2\rangle} = \frac{\langle y^2-x^2\rangle}{\langle x^2+y^2\rangle} \quad , \quad \epsilon_4^{(\mathbf{r})} \equiv -\frac{\langle r^4\cos(4\theta)\rangle}{\langle r^4\rangle}, \tag{1}$$

where (x,y) resp. (r,θ) are Cartesian resp. polar coordinates with the origin at the center of the overlap region, while the impact parameter between the two Pb nuclei is assumed to be along the x-direction. *Hint*: For $\epsilon_4^{(\mathbf{r})}$, it might be convenient to find the expression in Cartesian coordinates.

Solution:

On exercise 7 we computed $\mathrm{d}^2T_{AA}(\mathbf{b})/\mathrm{d}^2\mathbf{s} \equiv T_A(\mathbf{s}-\mathbf{b}/2)T_A(\mathbf{s}+\mathbf{b}/2)$ as function of the impact parameter (remember that the thickness function is nothing else than the integral of the density over the longitudinal direction).

In this exercise we want to compute the ellipticity and quadrangularity of the system. For the ellipticity we have

$$\epsilon_2^{(\mathbf{r})} = \frac{\langle y^2-x^2\rangle}{\langle x^2+y^2\rangle} = \frac{\int_{x,y}(y^2-x^2)T_A(\mathbf{s}-\mathbf{b}/2)T_A(\mathbf{s}+\mathbf{b}/2)dxdy}{\int_{x,y}(y^2+x^2)T_A(\mathbf{s}-\mathbf{b}/2)T_A(\mathbf{s}+\mathbf{b}/2)dxdy}$$

where, as usual, we will take the impact parameter along the x direction.

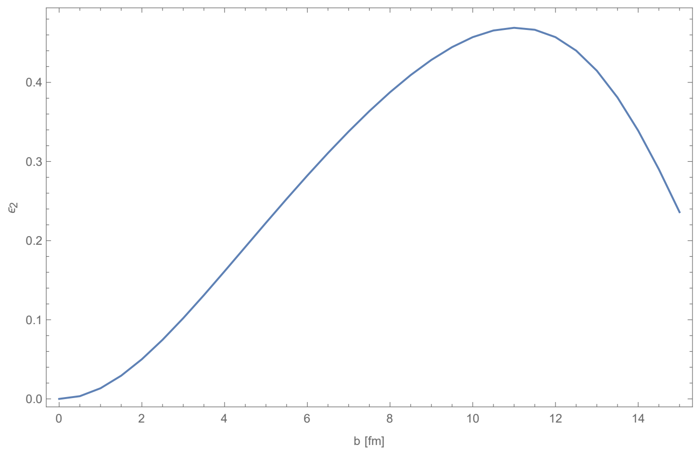

Figure 1: Ellipticity of the system as function of the impact parameter up to b=15 fm.

Notice in fig.(1) that the ellipticity is always positive (for the plotted range) which is because we "force" the system to be larger along the y direction. Also, for small impact parameters the elliptiicty is close to zero because the overlap area approaches quite well to a circle which has no ellipticity. Notice that the ellpiticty does not always grow with the impact parameter bur rather it reaches a maximum at around b=11 fm. Finally, it is nice to keep in mind a typical value for the ellipticity which we can estimate to be around 0.25.

For the quadrangularity we have (remember that $r^4 cos(4\theta) = x^4 + y^4 - 6x^2y^2$)

$$\epsilon_4^{(\mathbf{r})} = \frac{\int_{x,y}(-y^4 - x^4 + 6x^2y^2)T_A(\mathbf{s}-\mathbf{b}/2)T_A(\mathbf{s}+\mathbf{b}/2)dxdy}{\int_{x,y}(y^4 + x^4 + 2x^2y^2)T_A(\mathbf{s}-\mathbf{b}/2)T_A(\mathbf{s}+\mathbf{b}/2)dxdy}.$$

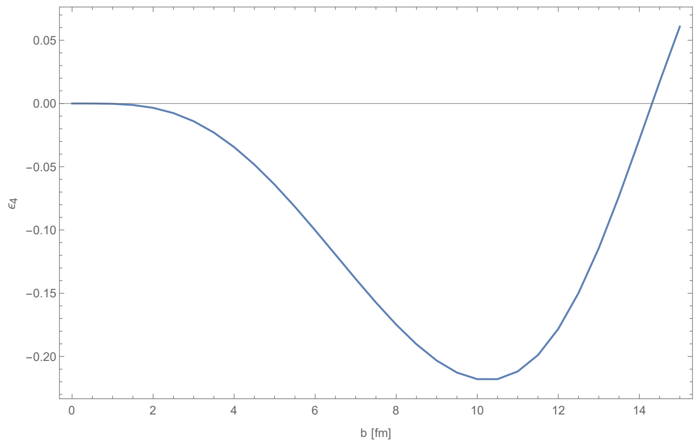

Figure 2: Quadrangularity of the system as function of the impact parameter up to b=15 fm.

The quadrangularity is most of the time negative (don't be scared) and it's values are quite smaller than the ellipticity which is normal since the system resembles an ellipse. If you want, you can go further and compute higher powers of ϵ_n and you should see that the odd ones give zero and the even ones become smaller as n increases.

11. Two-particle probability distributions

In this exercise, we assume that it is possible to find a class of events in which the impact-parameter direction Ψ_R is isotropically distributed, and such that at a fixed Ψ_R the single-particle azimuthal probability distribution is given by

$$p_1(\varphi|\Psi_R) = \frac{1}{2\pi}\big(1 + 2v_2\cos[2(\varphi - \Psi_R)]\big), \tag{2}$$

where v_2 is independent of Ψ_R. We want to investigate two-particle probability distributions, that describe the probability to have a first particle with azimuth φ_a and a second one with azimuth φ_b.

i. Azimuthally sensitive two-particle probability distribution

In the absence of (two-body) correlation, the two-particle probability distribution is simply the product of single-particle distributions: $p_2(\varphi_a, \varphi_b|\Psi_R) = p_1(\varphi_a|\Psi_R)\, p_1(\varphi_b|\Psi_R)$. Express this probability distribution in terms of the "pair angle" $\varphi^{\rm pair} \equiv \frac{1}{2}(\varphi_a + \varphi_b)$ and the angle difference $\Delta\varphi \equiv \varphi_a - \varphi_b$.

ii. Azimuthally insensitive two-particle probability distribution

a) In most experimental analyses, the two-particle distribution $p_2(\varphi^{\rm pair}, \Delta\varphi|\Psi_R)$ is integrated over $\varphi^{\rm pair}$ and averaged over Ψ_R. What do you obtain in that case? Let us denote $\mathfrak{p}(\Delta\varphi)$ the resulting distribution.

b) Can you *without calculation* tell how $\mathfrak{p}(\Delta\varphi)$ looks like if besides v_2 the single-particle probability distribution $p_1(\varphi|\Psi_R)$ also contains higher harmonics v_3, v_4, and v_5?

Plot your educated guess for $\mathfrak{p}(\Delta\varphi)$, multiplied by a factor 2π, for $-\frac{\pi}{2} \leq \Delta\varphi \leq \frac{3\pi}{2}$ with the values[1] $v_2 = 0.041$, $v_3 = 0.053$, $v_4 = 0.034$, and $v_5 = 0.012$ — and compare your results with figure 4 of the ALICE Collaboration article Phys. Rev. Lett. **107** (2011) 032302.[2]

iii. Azimuthally sensitive two-particle probability distribution revisited

Let us go back to question **i.**, again looking at $p_2(\varphi^{\rm pair}, \Delta\varphi|\Psi_R)$ resulting from the single-particle distribution (2). Focusing on the dependence on $\varphi^{\rm pair} - \Psi_R$ at fixed $\Delta\varphi$, how could you define "pair flow coefficients" to characterize it? Which of these harmonic coefficients are non-zero in the present case? Can you give their expressions? (Beware: a mistake is quickly made!)

Solution:

We know the single particle probability (or similarly, the single particle distribution, Question: what is the difference?) and since we assume that the two particle probability is just the product of the single particle probabilities then

$$p_2(\varphi^{\rm pair}, \Delta\varphi|\Psi_R) = \frac{1}{4\pi^2}(1 + 4v_2\cos(\Delta\varphi)\cos[2(\varphi_{pair} - \Psi_R)] + 2v_2^2(\cos(2\Delta\varphi) + \cos[4(\varphi_{pair} - \Psi_R)])$$

where we have used the change of variables as demanded.

It is common to be interested on the probability of finding two particles separated by a certain angle. In order to obtain that we have to average the probability over Ψ_R and integrate over φ_{pair} in the form

$$\mathfrak{p}(\Delta\varphi) = \frac{1}{2\pi}\int_{\Psi_R,\varphi_{pair}} \frac{1}{4\pi^2}(1+4v_2\cos(\Delta\varphi)\cos[2(\varphi_{pair}-\Psi_R)]+2v_2^2(\cos(2\Delta\varphi)+\cos[4(\varphi_{pair}-\Psi_R)])d\Psi_R d\varphi_{pair}$$

[1]... taken without any guarantee!

[2]You can also find the plot at `http://alice-publications.web.cern.ch/node/3879`.

This might look complicated but, if we rewrite the terms $\cos[2(\varphi_{pair} - \Psi_R])$ and $\cos[4(\varphi_{pair} - \Psi_R)]$ by using the expression $\cos(a - b) = \cos a \cos b + \sin a \sin b$ then the integrals become trivial since most of them vanish because $\int_0^{2\pi} \cos(2\Psi_R) = 0$ and similarly for the others.

All in all, what remains is

$$\mathfrak{p}(\Delta\varphi) = \frac{1}{2\pi}(1 + 2v_2^2 \cos(2\Delta\varphi))$$

If in the single particle probability there were higher harmonics $(v_3, v_4, ...)$ we would obtain similar results

$$\mathfrak{p}(\Delta\varphi) = \frac{1}{2\pi}(1 + 2v_2^2 \cos(2\Delta\varphi) + 2v_3^2 \cos(3\Delta\varphi) + ...)$$

where you can do the explicit calculation if you want (I recommend you to do that at least once).

The above probability is the probability of finding two particles with a certain angle difference. Since it is a probability you can see that if integrated over the angle difference it gives 1, this is a good test that shows that our calculation is not fully wrong.

For the next part we are asked to plot the probability we have obtained with certain values for the anistropic flow coefficients and to compare the result with a publication by ALICE.

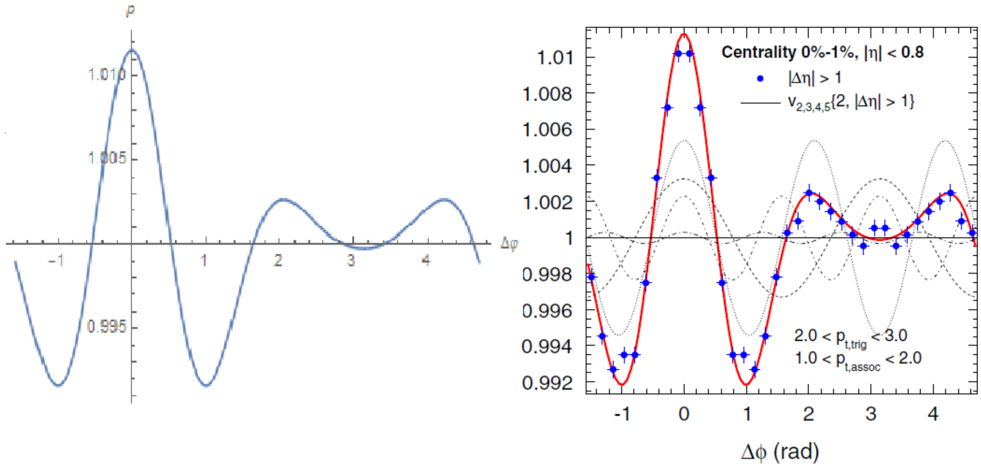

Figure 3: Two particle distribution compared to two particle azimuthal correlation measures in ALICE for central collisions.

Both probabilities look quite similar which means that the choice of the anisotropic flow coefficients is quite good. In the ALICE figure you can see the effect of each flow coefficient by itself. Notice that the largest is v_3 and not v_2 that is because the collisions are central. Also notice that in general two particles will have an angle difference of zero.

Last thing we have to do is to look again at the two particle distribution

$$p_2(\varphi^{\text{pair}}, \Delta\varphi|\Psi_R) = \frac{1}{4\pi^2}(1 + 4v_2 \cos(\Delta\varphi) \cos[2(\varphi_{pair} - \Psi_R)] + 2v_2^2(\cos(2\Delta\varphi) + \cos[4(\varphi_{pair} - \Psi_R)]).$$

In this case we assume that the angle difference is constant and we want to characterize the distribution with respect to $\varphi^{\text{pair}} - \Psi_R$. This means that we have to Fourier expand the the two particle distribution as it was done for the single particle distribution but this time the expansions is respect to $\varphi^{\text{pair}} - \Psi_R$. That means the following,

$$\begin{aligned} p_2(\varphi^{\text{pair}}|\Delta\varphi, \Psi_R) &= \frac{1}{4\pi^2}(1 + 4v_2\cos(\Delta\varphi)\cos[2(\varphi_{pair} - \Psi_R)] + 2v_2^2(\cos(2\Delta\varphi) + \cos[4(\varphi_{pair} - \Psi_R)]) \\ &= \alpha(1 + 2v_2'\cos[2(\varphi_{pair} - \Psi_R)] + 2v_n'\cos[n(\varphi_{pair} - \Psi_R)]) \end{aligned} \tag{3}$$

where v' are the new flow coefficients and we are at fixed $\Delta\varphi$ and α is a normalization constant.

We can easily see that the only two coefficients that would be different than zero would be v_2' and v_4'.

We can obtain α by averaging the probability over Ψ_R and integrating over φ_{pair} which gives

$$\alpha = \frac{1 + 2v_2^2\cos(\Delta\varphi)}{2\pi}.$$

We need to compute the expressions of the new flow coefficients. For a Fourier series they are computed as

$$v_n' \equiv \frac{\int p_2 \cos(n(\varphi_{pair} - \Psi_R))}{\int p_2} = \frac{2\pi\alpha v_n'}{2\pi\alpha} = v_n'$$

where we integrate over φ_{pair} and average over Ψ_R.

Hence, we obtain

$$v_2' = 2v_2\cos(\Delta\varphi) \qquad v_4' = v_2^2$$