Tutorial sheet 4

Discussion topic: Anisotropic transverse flow: how is it quantified? what are the necessary ingredients for its appearance?

8. A bizarre exercise

In this exercise — whose hidden content will hopefully become clearer in the lecture on May 26 — you are to consider functions of a variable z, whose Taylor expansion involves graphic coefficients (•), (• •), (•••) and so on. One can multiply those coefficients using "normal" rules, such that each bullet • remains confined within its original subgraph, and that different subgraphs do not merge: for instance (•)^2 = (•)(•) is not the same as (• •).

i. Let $f(z) \equiv$ (•)$\,z\,+$ (• •)$\,\dfrac{z^2}{2!} +$ (•••)$\,\dfrac{z^3}{3!} +$ (••••)$\,\dfrac{z^4}{4!} + \cdots$.

Using the Taylor expansion of e^x for small x, compute $\exp[f(z)]$ to order $\mathcal{O}(z^4)$.

ii. Consider now graphs consisting of one, two, three, four, … bullets, that are now no longer enclosed in "connected groups": •, • •, •••, ••••, …. All bullets of a given (non-connected) graph are supposed to be different, i.e. can be designated with different labels, as e.g. (1,2,3) for the bullets of •••.

a) For each non-connected graph with $n = 2$, 3 or 4 bullets, find all possible ways of "decomposing" the graph by regrouping all its bullets in non-overlapping connected groups of $1 \leq m \leq n$ bullets. For $n = 1$, i.e. •, there is a single possibility: (•).

Hint: Starting with $n = 3$, there might be different groupings with the same "topology", say for instance (for $n = 3$) with one pair and one single bullet: you may regroup these groupings — but do not forget their multiplicity, i.e. how many groupings have that topology.

b) Compare your "rewritings" of the disconnected graphs • •, •••, •••• in terms of connected subgraphs with the coefficients of z^2, z^3, z^4 of the function $\exp[f(z)]$ in question **i.** What do you notice?[1]

9. Free-streaming expansion

Consider a two-dimensional system of free-streaming particles. At time $t = 0$, the geometry of the system is characterized by its typical (squared) size $R^2 \equiv \langle x^2 + y^2 \rangle$ and its "ellipticity"

$$\epsilon_2^{(\mathbf{r})} \equiv \frac{\langle y^2 - x^2 \rangle}{\langle x^2 + y^2 \rangle} = \epsilon_2^{(\mathbf{r})}(0), \tag{1}$$

where (x, y) are Cartesian coordinates — with the origin at the center of the system — and $\langle \cdots \rangle$ denote an average which need not be further specified. The particle momenta and velocities at $t = 0$ are distributed isotropically, with $\langle v_x^2 \rangle = \langle v_y^2 \rangle \equiv \langle \mathbf{v}^2 \rangle / 2$

As time goes by, the system evolves, and in particular its typical size and ellipticity are changing: compute $\epsilon_2^{(\mathbf{r})}(t)$ at time t.

Hint: Begin with the time-dependence of $\langle x^2 \rangle$ and $\langle y^2 \rangle$. In the end, $\epsilon_2^{(\mathbf{r})}(t)$ can be expressed in terms of $\epsilon_2^{(\mathbf{r})}(0)$, R^2 and $\langle \mathbf{v}^2 \rangle$ — and naturally t.

[1] You are free to go to graphs with 5 or 6 bullets if you cannot sleep at night!