Solutions to sheet 4

Discussion topic: Anisotropic transverse flow: how is it quantified? what are the necessary ingredients for its appearance?

It is quantified by the use of the anisotropic flow coefficients which are the coefficients of a Fourier series of the particle distribution in momentum space (in the transverse direction). The necessary ingredients to have flow is basically that the initial state shape is not a circle but rather an ellipse or something similar. Therefore, since particles may rescatter this generates anisotrpic flow.

8. A bizarre exercise

In this exercise — whose hidden content will hopefully become clearer in the lecture on May 26 — you are to consider functions of a variable z, whose Taylor expansion involves graphic coefficients $\odot$, $(\bullet\bullet)$, $(\because)$ and so on. One can multiply those coefficients using "normal" rules, such that each bullet $\bullet$ remains confined within its original subgraph, and that different subgraphs do not merge: for instance $\odot^2 = \odot\odot$ is not the same as $(\bullet\bullet)$.

i. Let $f(z) \equiv \odot z + (\bullet\bullet)\frac{z^2}{2!} + (\because)\frac{z^3}{3!} + (::)\frac{z^4}{4!} + \cdots$.

Using the Taylor expansion of e^x for small x, compute $\exp[f(z)]$ to order $\mathcal{O}(z^4)$.

Solution: By computing the Taylor expansion of the exponential function ($e^x \approx 1 + x + x^2/2 + ...$) using f(z) we obtain:

$$e^{f(z)} = 1+\odot z+\frac{1}{2}(\odot^2+(\bullet\bullet))z^2+\frac{1}{6}(\odot^3+3\odot(\bullet\bullet)+(\because))z^3+\frac{1}{24}(\odot^4+6\odot^2(\bullet\bullet)+4\odot(\because)+3(\bullet\bullet)^2+(::))z^4+\mathcal{O}(z^5) \tag{1}$$

ii. Consider now graphs consisting of one, two, three, four, … bullets, that are now no longer enclosed in "connected groups": $\bullet$, $\bullet\,\bullet$, $\because$, $::$, …. All bullets of a given (non-connected) graph are supposed to be different, i.e. can be designated with different labels, as e.g. (1,2,3) for the bullets of $\because$.

a) For each non-connected graph with $n = 2$, 3 or 4 bullets, find all possible ways of "decomposing" the graph by regrouping all its bullets in non-overlapping connected groups of $1 \leq m \leq n$ bullets. For $n = 1$, i.e. $\bullet$, there is a single possibility: $\odot$.

Hint: Starting with $n = 3$, there might be different groupings with the same "topology", say for instance (for $n = 3$) with one pair and one single bullet: you may regroup these groupings — but do not forget their multiplicity, i.e. how many groupings have that topology.

b) Compare your "rewritings" of the disconnected graphs $\bullet\,\bullet$, $\because$, $::$ in terms of connected subgraphs with the coefficients of z^2, z^3, z^4 of the function $\exp[f(z)]$ in question **i.** What do you notice?[1]

Solution:

For n=1 the only option is $\bullet = \odot$.

For n=2 we have $\bullet\,\bullet = (\bullet\bullet) + \odot\,\odot$.

For n=3 and n=4 we have (see figures)

Comparing the coefficients of the expansion of $e^{f(z)}$ (see eq.(1)) we notice that the power z^n corresponds to the n-th disconnected graph decomposition.

[1] You are free to go to graphs with 5 or 6 bullets if you cannot sleep at night!

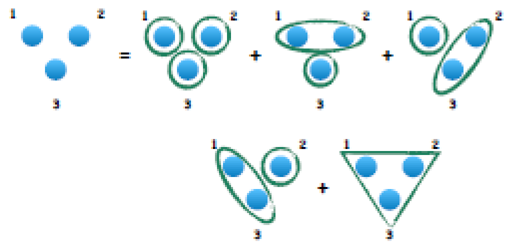

Figure 1: Three particle correlations.

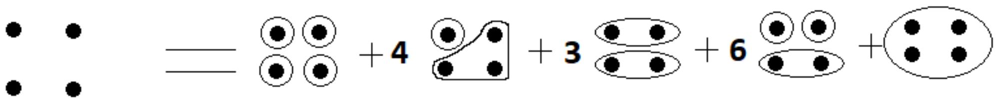

Figure 2: Four particle correlations.

You may ask what have we been doing. Basically, we are writing the n-th particle distribution function. Let us see it for some values of n.

For n=1 the single particle distribution function is: $f(x,p) \propto e^{-E/T}$ where we assumed the distribution to be a Boltzmann distribution. The distribution is the probability of finding a particle at position x and momentum p. For n=2 the two particle distribution function can be written as: $f(x_1,p_1,x_2,p_2) = f(x_1,p_1)f(x_2,p_2) + f^c(x_1,x_2,p_1,p_2)$. Where the first part is just the product of the single particle distributions which includes no correlations between the particles and the second part is the correlated two particle function which precisely accounts for the correlations between two particles. The superindex c refers to the correlation part.

For n=3 we can write (let me define $f_i \equiv f(x_i,p_i)$ and similarly for f_{ij})

$$f(x_1,p_1,x_2,p_2,x_3,p_3) = f_1f_2f_3 + f^c_{12}f_3 + f^c_{13}f_2 + f^c_{23}f_1 + f^c_{123}$$

Why are particles correlated with each other? Due to different physical processes. Some of them are: decay products, quantum effects, momentum conservation or in general any conservation law , Couloumb interactions, etc.

9. Free-streaming expansion

Consider a two-dimensional system of free-streaming particles. At time $t = 0$, the geometry of the system is characterized by its typical (squared) size $R^2 \equiv \langle x^2 + y^2 \rangle$ and its "ellipticity"

$$\epsilon_2^{(\mathbf{r})} \equiv \frac{\langle y^2 - x^2 \rangle}{\langle x^2 + y^2 \rangle} = \epsilon_2^{(\mathbf{r})}(0), \tag{2}$$

where (x, y) are Cartesian coordinates — with the origin at the center of the system — and $\langle \cdots \rangle$ denote an average which need not be further specified. The particle momenta and velocities at $t = 0$ are distributed isotropically, with $\langle v_x^2 \rangle = \langle v_y^2 \rangle \equiv \langle \mathbf{v}^2 \rangle / 2$

As time goes by, the system evolves, and in particular its typical size and ellipticity are changing: compute $\epsilon_2^{(\mathbf{r})}(t)$ at time t.

Hint: Begin with the time-dependence of $\langle x^2 \rangle$ and $\langle y^2 \rangle$. In the end, $\epsilon_2^{(\mathbf{r})}(t)$ can be expressed in terms of $\epsilon_2^{(\mathbf{r})}(0)$, R^2 and $\langle \mathbf{v}^2 \rangle$ — and naturally t.

Solution: The ellipticity of a system can be written as

$$\epsilon_2^r = \frac{- < x^2 - y^2 >}{< x^2 + y^2 >} \tag{3}$$

where the integral runs over position and momentum.

When a system freely streams one can write

$$x(t) = x_o + v_x t$$

$$y(t) = y_o + v_y t.$$

The combination of the above expressions yields

$$\epsilon_2^r(t) = \frac{- < x_o^2 - y_o^2 + v_x x_o t - v_y y_o t + v_x^2 t^2 - v_y^2 t^2 >}{< x_o^2 + y_o^2 + v_x x_o t + v_y y_o t + v_x^2 t^2 + v_y^2 t^2 >}, \tag{4}$$

which can be simplified if one assumes that the odd powers in the velocity vanish (since the integral over momentum would give zero) and one also uses momentum isotropy ($< v_x >^2 = < v_y >^2 = v^2/2$) is used. Thus, the expression becomes

$$\epsilon_2^r(t) = \frac{y_o^2 - x_o^2}{x_o^2 + y_o^2 + v^2 t^2} = \frac{\epsilon_2(0)}{1 + \frac{v^2 t^2}{r_o^2}}, \tag{5}$$

which tells us that an initial anisotropy decreases over time as we expected for a system that expands at the same velocity in all directions.

As an exercise you can try to compute now other anisotropies ($\epsilon_4^r(t)$ for example). You should see that for small times all anisotropies follow the same pattern.