Solutions to tutorial sheet 2

Discussion topic: What are the center-of-mass energy per nucleon pair, the longitudinal rapidity, the pseudorapidity, the transverse mass?

$c = 1$ throughout the exercise sheet.

4. Rapidity of the nuclei

At the Large Hadron Collider, there have been Pb–Pb collisions at $\sqrt{s_{\rm NN}} = 2.76$ TeV and at 5.02 TeV. What is the rapidity $y_{\rm Pb}$ of the Pb beams in those collisions?
More generally, use your favorite plotting software to plot $y_{\rm Pb}$ vs. $\sqrt{s_{\rm NN}}$ for $\sqrt{s_{\rm NN}} \leq 10$ TeV.
Hint: In case you need it, the mass of a lead nucleus was given in exercise **1.**

Solution:

One way of writing the rapidity is

$$y = atanh(v_z). \tag{1}$$

We just need to express the velocity as a function of the center of mass energy. Remember from QM lectures that

$$E = m\gamma = \frac{1}{2}\sqrt{s_{NN}} \qquad \rightarrow \quad \gamma = \frac{\sqrt{s_{NN}}}{2m} \quad \rightarrow \quad v_z^2 = 1 - \gamma^{-2} = 1 - \left(\frac{2m}{\sqrt{s_{NN}}}\right)^2 \tag{2}$$

where m is the mass of a nucleon which you can either compute as $\frac{m_{Pb}}{A} = 0.937 GeV$ or take the mass of a neutron (940 MeV).

Therefore, the rapidity becomes

$$y = atanh\left(\sqrt{1 - \left(\frac{2m}{\sqrt{s_{NN}}}\right)^2}\right) \tag{3}$$

Notice that the rapidity of a Pb beam or of one nucleon of a Pb nucleus are the same (they have the same velocity).

At $\sqrt{s_{\rm NN}} = 2.76$ TeV the rapidity is y = 8 and at 5.02 TeV it is y = 8.6 .

Question: Will the pseudorapidity for the energies 2.76 TeV and 5.02 TeV be too different from the rapidity? Would the same happen for much lower energies?

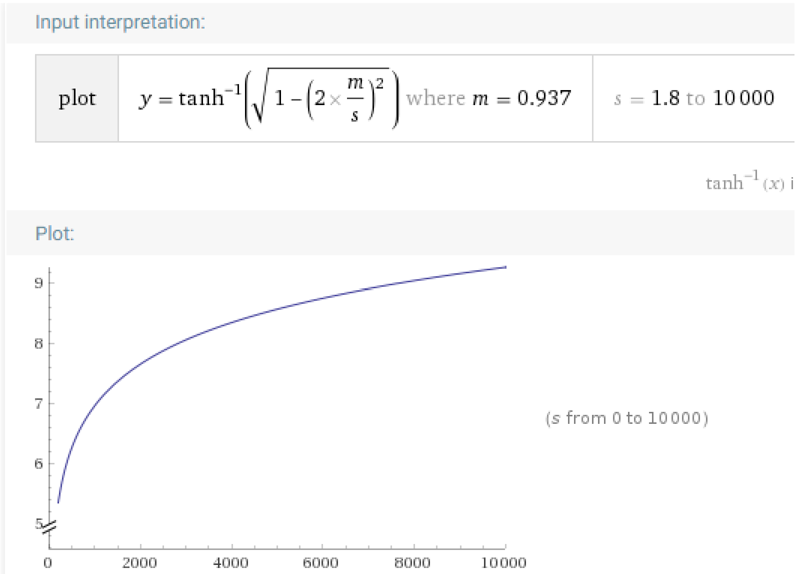

Figure 1: Rapidity (vertical axis) as function of $\sqrt{s_{NN}}$ in GeV.

5. Center-of-momentum energy and rapidity per nucleon pair

Again at the LHC, the beams of nuclei — here denoted A_1 and A_2 — circulating in opposite directions in the accelerator have the same "rigidity" $|\vec{p}|/q$ with $\vec{p}$ and q the (total) momentum and electric charge of the nucleus.
Bonus question: can you see why? Nuclei are accelerated with electromagnetic fields. Therefore, the force that accelerates them only acts on the charged nucleons (protons). The force is used to make the nuclei rotate (just like the moon and the earth). Therefore, $r = p/qB$ where r is the radius of the LHC, B the magnetic field used. This implies that p/q is fixed by the experimental set up.

i. Show that the center-of-mass energy per nucleon pair for an ultrarelativistic $A_1 + A_2$ collision is given by

$$\sqrt{s_{\rm NN}} \simeq 2|\vec{p}_p|\sqrt{\frac{Z_1 Z_2}{A_1 A_2}} \tag{4}$$

where Z_1, Z_2 are the proton numbers of the colliding nuclei and $|\vec{p}_p|$ the momentum in the lab frame of a proton with the same rigidity as the nuclei.
What is the value of $\sqrt{s_{\rm NN}}$ for p–Pb collisions with protons of energy 6.5 TeV?

Solution:

Since the rigidity is fixed then the ratio p/q for a nucleus is also fixed. The momentum in the transverse direction is zero. The energy and momentum of the nucleus A_1 is

$$p_{1,z} = |\vec{p}_p|Z_1 \qquad E_1 = \sqrt{m_n^2 A_1^2 + |\vec{p}_p|^2 Z_1^2} \approx |\vec{p}_p|Z_1 \tag{5}$$

were we have used the fact that the particles are ultrarelativistic. For the second nucleus we find

$$p_{2,z} = -|\vec{p}_p|Z_2 \qquad E_2 = \sqrt{m_n^2 A_2^2 + |\vec{p}_p|^2 Z_2^2} \approx |\vec{p}_p|Z_2. \tag{6}$$

The center of mass energy will be (following the lecture notes)

$$s = (E_1 + E_2) - (\vec{p_1} + \vec{p_2})^2 = E_1^2 + E_2^2 + 2E_1E_2 - p_1^2 - p_2^2 - 2\vec{p_1}\vec{p_2} \tag{7}$$

Combining all the above equations we find

$$s = 4|\vec{p}_p|^2 Z_1 Z_2 \tag{8}$$

The center of mass energy per nucleon pair is defined as

$$\sqrt{s_{NN}} \equiv \frac{\sqrt{s}}{\sqrt{A_1 A_2}} = 2|\vec{p}_p|\sqrt{\frac{Z_1 Z_2}{A_1 A_2}} \tag{9}$$

Notice that if the nuclei are not relativistic the result becomes more complicated.
The value for a proton-lead collisions with protons energey of 6.5 TeV is

$$\sqrt{s_{NN}} = 2|\vec{p}_p|\sqrt{\frac{Z_1 Z_2}{A_1 A_2}} = 2 \cdot 6.5TeV\sqrt{\frac{82}{208}} = 8.16TeV \tag{10}$$

ii. Show that the corresponding center-of-momentum frame of nucleon–nucleon collisions has the rapidity

$$|y_{\mathrm{NN}}| = \frac{1}{2}\ln\frac{Z_1 A_2}{A_1 Z_2} \tag{11}$$

in the lab frame. What value does this give for p–Pb collisions at the LHC?

Note that the c.m. frame of nucleon–nucleon collisions considered here actually differs from the true c.m. frame of the $A_1 + A_2$ collision, whose rapidity can be computed with the help of the following exercise.

Solution:

The momentum of a nucleon in the laboratory frame is

$$p^l_{n,1} = |\vec{p}_p|\frac{Z_1}{A_1} = m_n \sinh y^l_{n_1} \qquad p^l_{n,2} = -|\vec{p}_p|\frac{Z_2}{A_2} = m_n \sinh y^l_{n,2} \tag{12}$$

In the c.m. frame we find

$$p^{cm}_{n,1} + p^{cm}_{n,2} = 0 \quad \to m_n \sinh y^{cm}_{n,1} + m_n \sinh y^{cm}_{n,2} = 0 \quad \to \sinh y^{cm}_{n,1} = \sinh -y^{cm}_{n,2} \tag{13}$$

And, the rapidity in the center of mass frame is

$$y^{cm}_{n,1} = y^l_{n,1} - y^{cm} \tag{14}$$

Combining the equations 13 and 14 we find

$$y^{cm} = \frac{y^l_{n,1} + y^l_{n,2}}{2} \tag{15}$$

(which is the first equation of the next exercise if the masses are equal). The rapidities in the laboratory frame are know due to eq.(12), we just rewrite them as

$$y^l_{n,1} = asinh(|\vec{p}_p|\frac{Z_1}{m_n A_1}) \qquad y^l_{n,2} = -asinh(|\vec{p}_p|\frac{Z_2}{m_n A_2}). \tag{16}$$

Formally we have solved the problem if we insert eq. (16) into eq.(15). But this has not the same form as what we are asked. This is because we have to use the fact that particles are ultrarelativistic. From here you can use any program you like to make it compute this for you but we will do it by hand.

All we have to use is the relation

$$asinh(x) = ln(x(1 + \sqrt{1 + \frac{1}{x^2}})$$

where, for our case,

$$x = |\vec{p}_p| \frac{Z_1}{m_n A_1} >> 1$$

and therefore

$$y^l_{n,1} = asinh(|\vec{p}_p| \frac{Z_1}{m_n A_1}) \approx ln(2|\vec{p}_p| \frac{Z_1}{m_n A_1}) \qquad y^l_{n,1} = -asinh(|\vec{p}_p| \frac{Z_1}{m_n A_1}) \approx -ln(2|\vec{p}_p| \frac{Z_1}{m_n A_1}).$$

Eq.(14) becomes

$$|y^{cm}| = \frac{1}{2} \ln \frac{Z_1 A_2}{A_1 Z_2}$$

For a proton-lead collision at the LHC we find

$$|y^{cm}| = \frac{1}{2} \ln \frac{208}{82} = 0.47$$

The lab and the cm frames do not coincide!

6. Center-of-momentum rapidity in asymmetric collisions

Consider the collision $a + b \to \cdots$ where a and b have respective masses m_a, m_b and rapidities (in the lab frame) y_a, y_b. Show that the rapidity of the center-of-momentum frame of the system in the lab frame is

$$|y_{\text{c.m.}}| = \frac{y_a + y_b}{2} + \frac{1}{2} \ln \frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{y_b} + m_b e^{y_a}}. \tag{17}$$

Solution: We saw in the lecture that

$$p_z = m_T \sinh y \qquad m_T^2 = m^2 + p_x^2 + p_y^2 = m^2. \tag{18}$$

In the center of momentum frame (c.m.) we know that

$$p_{z,a} + p_{z,b} = 0. \tag{19}$$

Therefore, in the CM frame,

$$m_a \sinh y_{a,c.m.} + m_b \sinh y_{b,c.m.} = 0 \tag{20}$$

where $y_{a,c.m.}$ is the rapidity of particle a viewed in the CM frame.

Since rapidities are additive,

$$y_{a,c.m.} = y_a - y_{c.m.}.$$

Hence,

$$m_a \sinh(y_a - y_{c.m.}) + m_b \sinh(y_b - y_{c.m.}) = 0 \tag{21}$$

which has to be valid for any y_a and y_b. You can easily see that, if $m_a = m_b$ the solution becomes $y_{c.m.} = \frac{y_a+y_b}{2}$. But in general the masses are different. Expressing the hyperbolic sines in their exponential form and performing some algebra leads us to (try to do the algebra if you did not do it)

$$y_{\text{c.m.}} = \frac{y_a + y_b}{2} + \frac{1}{2} \ln \frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{y_b} + m_b e^{y_a}}$$

As you can see, using the rapidity offers the advantage that changing frames is much easier.