Solutions to sheet 11

Discussion topic: Hadrochemistry: what is strangeness enhancement? the idea of the statistical model of hadron production?

Throughout the exercise sheet, a system of units such that the constants c, $\hbar$ and k_B equal 1 is used.

22. Kaon-to-pion ratio

Consider a system of u, d, s quarks and their antiquarks, whose respective amounts are denoted N_u, N_d, N_s and so on. This system is supposed to "hadronize" fully into a system of pions (π^+, π^-, π^0) and kaons (K^+, K^-, K^0, $\overline{K^0}$), such that equal amounts of each species in a family are produced: $N_{\pi^+} = N_{\pi^-} = N_{\pi^0}$ and a similar equality among the four kaon species.

Express the ratio N_{K^+}/N_{π^+} as a function of the numbers of quarks and antiquarks.

Solution:

We have three pions and four kaons, which quark content are:

$$\pi^+ = u\bar{d} \quad \pi^- = d\bar{u} \quad \pi^0 = u\bar{u} + d\bar{d} \quad K^+ = u\bar{s} \quad K^- = s\bar{u} \quad K^0 = d\bar{s} \quad \bar{K}^0 = s\bar{d}.$$

Since the kaon amounts per specie are the same (N_K), and equivalently for pions (N_π), we can count the number of strange and anti-strange quarks, which is

$$N_s = N_{\bar{K}^0} + N_{K^-} = N_K/2 \qquad N_{\bar{s}} = N_{K^0} + N_{K^+} = N_K/2,$$

and similarly for up quarks we get

$$N_u = N_{K^+} + N_{\pi^+} + \frac{N_{\pi^0}}{2} = N_K/4 + N_\pi/2$$

where the factor $1/2$ is because half of the π^0 will have an up quark. Similar equations are obtained for the remaining quarks.

All in all we get,

$$N_s = N_{\bar{s}} = N_K/2 \qquad N_u = N_{\bar{u}} = N_d = N_{\bar{d}} = N_K/4 + N_\pi/2.$$

The ratio N_{K^+}/N_{π^+} will be

$$N_{K^+}/N_{\pi^+} = \frac{\frac{1}{2}N_s}{\frac{1}{3}(2N_u - N_s)}$$

which you can rewrite in many forms. Notice that if all quarks are equally abundant then we get $N_{K^+}/N_{\pi^+} = \frac{3}{2}$ i.e. there are more kaons than pions. In general that is not the case since light quarks are more abundant.

23. Canonical suppression factor

Fill the gaps in the lecture: on slides 27–30 of the "Hadrochemistry" lecture, a number of technical calculations were left aside. Starting from the single-particle partition functions z_+, z_-, z_s of K^+, K^- and a hadron species with strangeness $S = s$ (hereafter: "hyperons"), compute the average number $\langle N_s \rangle$

of hyperons in the grand canonical and the canonical ensembles and deduce the canonical suppression factor of the hyperons.

Solution:

The results of this exercise are on the slides 27-30. What we want to do is to make the calculations which are not there.

Let us being with the grand canonical ensemble. The grand partition function reads as

$$\Omega = \sum_{N_+, N_-, N_s} Z(N_+, N-, Ns)\lambda^{N_+ - N_- + sN_s}$$

where λ is the fugacity and Z the partition function. For an ideal gas (slide 23) we have

$$Z(N_+, N-, Ns) = \frac{z_+^{N_+}}{N_+!}\frac{z_-^{N_-}}{N_-!}\frac{z_s^{N_s}}{N_s!}$$

Since we are in the g.c. ensemble the system can interchange particles with the reservoir. Therefore (the sums run from 0 to infinite),

$$\Omega = \sum_{N_+, N_-, N_s} \frac{z_+^{N_+}}{N_+!}\frac{z_-^{N_-}}{N_-!}\frac{z_s^{N_s}}{N_s!}\lambda^{N_+ - N_- + sN_s} = e^{\lambda z_+ + \frac{z_-}{\lambda} + z_s\lambda^s},$$

where

$$\frac{1}{\Omega}\frac{z_+^{N_+}}{N_+!}\frac{z_-^{N_-}}{N_-!}\frac{z_s^{N_s}}{N_s!}\lambda^{N_+ - N_- + sN_s}$$

is the probability of having N_+, N_- and N_s particles in the system.

If we want to obtain $< N_s >$ in the g.c. ensemble we have to compute the probability of having N_s particles times N_s. That gives,

$$< N_s >= \frac{1}{\Omega}\sum_{N_+, N_-, N_s} \frac{z_+^{N_+}}{N_+!}\frac{z_-^{N_-}}{N_-!}\frac{z_s^{N_s}}{N_s!}\lambda^{N_+ - N_- + sN_s} N_s = \frac{1}{\Omega} e^{\lambda z_+ + \frac{z_-}{\lambda}} \sum_{N_s=0}^{N_s=\infty} \frac{z_s^{N_s}}{N_s!}\lambda^{sN_s} N_s =$$

$$= \frac{1}{\Omega} e^{\lambda z_+ + \frac{z_-}{\lambda}} \sum_{N_s=0}^{N_s=\infty} \frac{(z_s\lambda^s)^{N_s - 1}}{(N_s - 1)!} z_s\lambda^s = z_s\lambda^s$$

$$< N_s >_{g.c.} = z_s\lambda^s$$

For the canonical ensemble we have a difference, strangeness is conserved. We also assume that total strangeness is zero which means that

$$N_- = N_+ + sN_s$$

Similarly as before we can write the partition function as

$$Z(N_+, N-, Ns) = \frac{z_+^{N_+}}{N_+!}\frac{z_-^{N_-}}{N_-!}\frac{z_s^{N_s}}{N_s!} = \frac{z_+^{N_+}}{N_+!}\frac{z_-^{N_+ + sN_s}}{(N_+ + sN_s)!}\frac{z_s^{N_s}}{N_s!}$$

bu now we have used the relation between particles abundancies. You can choose any of the two independent variables.

Now we can sum as before, but before, a useful thing,

$$I_n(2x) = \sum_0^\infty \frac{x^{2N+n}}{N!(N+n)!}$$

$$Z = \sum_{N_+,N_s} Z(N_+, N-, Ns) = \sum_{N_s} \frac{z_s^{N_s}}{N_s!} \sum_{N_+} \frac{(z_+z_-)^{N_+ + sN_s/2}}{N_+!(N_+ + sN_s)!} (z_-/z_+)^{sN_s/2} =$$

$$= \sum_{N_s} \frac{z_s^{N_s}}{N_s!} (z_-/z_+)^{sN_s/2} I_{sNs}(2\sqrt{z_+z_-})$$

The above expression is complicated to sum. Anticipating a very smaller number $Ns << 1$, we keep only the terms Ns = 0 and Ns = 1 in the partition function and assume that the latter is much smaller than the former (from slide 29). We find,

$$Z = I_o(2\sqrt{z_+z_-}) + (z_-/z_+)^{sN_s/2} z_s I_s(2\sqrt{z_+z_-}) \approx I_o(2\sqrt{z_+z_-})$$

As we did before in the g.c. ensemble, we can compute now $< N_s >$ in the canonical ensemble, which leads to

$$< N_s >_{can.} = \frac{\sum_{N_s} \frac{z_s^{N_s}}{N_s!} \sum_{N_+} \frac{(z_+z_-)^{N_+ + sN_s/2}}{N_+!(N_+ + sN_s)!} (z_-/z_+)^{sN_s/2} N_s}{Z}$$

One again the sum is too complicated so we compute it for Ns=0 and Ns=1 but for the first case it vanishes, hence,

$$< N_s >_{can.} = \frac{(z_-/z_+)^{s/2} z_s I_s(2\sqrt{z_+z_-})}{I_o(2\sqrt{z_+z_-})}$$

Finally, the canonical suppression factor of the hyperons becomes

$$\frac{< N_s >_{can.}}{< N_s >_{g.c.}} = \frac{(z_-/z_+)^{s/2} I_s(2\sqrt{z_+z_-})}{I_o(2\sqrt{z_+z_-})\lambda^s}$$

And, in the limit where there are few hyperons compared to Kaons (slide 28) the fugacity becomes $\lambda = \sqrt{z_-/z_+}$ which leads to

$$\frac{< N_s >_{can.}}{< N_s >_{g.c.}} = \frac{I_s(2\sqrt{z_+z_-})}{I_o(2\sqrt{z_+z_-})}$$

which is smaller than one always and it becomes even larger as s increases.

24. Ratios of light nuclei in the statistical model

i) Calculate the ratio d : ^{3}He : ^{4}He of the yields of deuterons, ^{3}He, and ^{4}He nuclei in the statistical model for $T = 156.5$ MeV and vanishing chemical potentials μ_B and μ_{I_3} (nuclear masses: $m_d = 1.8756$ GeV, $m_{^3\mathrm{He}} = 2.8084$ GeV, $m_{^4\mathrm{He}} = 3.7274$ GeV).

Solution:

Important note: It was forgotten to specify that the spins of the particles are 1, 1/2 and 0 respectively.

From the slides (slide 20) we can read the particle number density for a quantum gas. This is a generalization to what we did on sheet 9 where we computed the same but without taking into an account quantum effects.

The density reads as,

$$n = \frac{gTm^2}{2\pi^2} \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k} \lambda^k K_2 \left(\frac{km}{T} \right),$$

where $\lambda = e^{(\mu_B B + \mu_S S + \dots)/T}$ is the fugacity which is zero for our case. You can see that the first term of the sum is precisely what we deduced for a classical gas.

From here one just simply computes the expression numerically. The sum can be safely stopped after a few k's. Remember that the degeneracy factors are 3,2 and 1 respectively.

The ratios become,

$$n_d / n_{He^3} \approx 333 \qquad n_{He^3} / n_{He^4} \approx 476$$

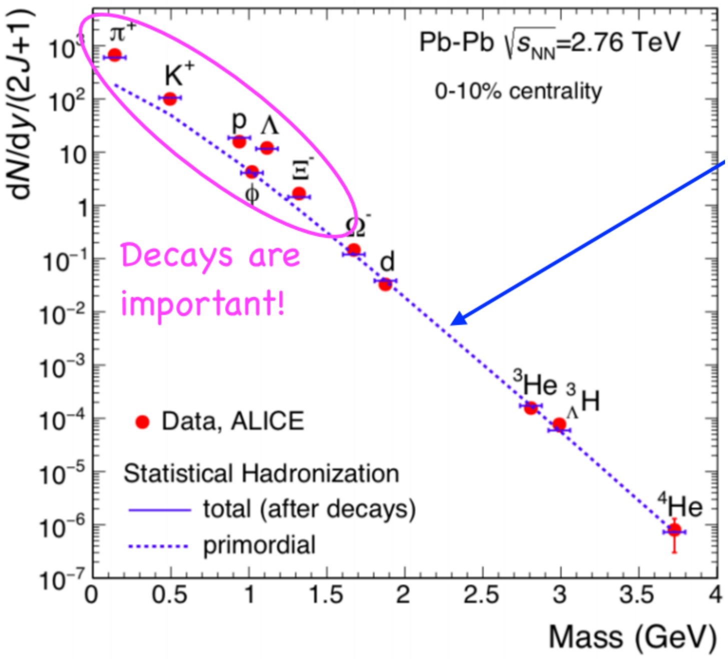

Figure 1: Particle distribution (primordial and total) for Pb-Pb at 2.76 TeV

ii) Plot the particle density n per spin degree of freedom as a function of the mass m for $T = 156.5$ MeV: a) taking quantum statistics into account; b) in the Boltzmann approximation; c) in the Boltzmann approximation using the leading order of the large-argument approximation of the modified Bessel functions of the second kind [see exercise **19.iii)**]. What do you conclude?

Solution:

Here we will plot the particle density as a function of the mass. But we will use three options, the expression above there quantum statistics are accounted for, the Boltzmann approximation which is the case k=1, and the expansion of a Bessel function as we did in sheet 9 keeping only the leading term.

Q: What do you conclude?

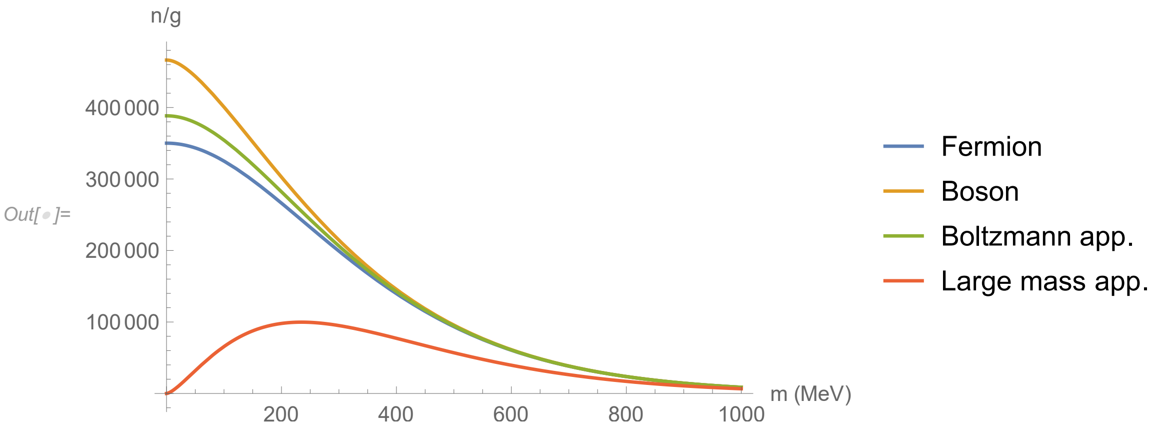

Figure 2: particle density (in MeV^3) for different systems.

One should see that at large mass values all approaches basically converge, that means that the typical Boltzmann distribution works well.

At low mass values things become funnier. The large mass approximations fails strongly which makes sense. The Boltzmann distribution lies in the middle between the fermionic and bosonic ones. Notice that the bosonic is the largest which is the typical boson condensation effect and the opposite for fermions. As usual quantum effects are only important at low m/T values which is not the case for the three particles we considered before which are considerably heavy (this is why we could stop the sum without problems).