Solution to sheet 10

Discussion topic: Femtoscopy: what are "HBT radii"? How are they extracted from experimental data? How are they affected by the expansion of the fireball?

HBT radii measure the size of the system at first approximation. Actually the measure something smaller than that, the region of homogeneity. That is, the region in space where the emitted particles have some correlation. By logic this region can not be larger than the size of the source but it can be far smaller. If you are lucky they will be quite similar hence the HBT radii will give a good estimation of the system size.

Experimentally, the two-particle correlation function is defined as the ratioC(q)=A(q)/B(q), whereA(q)is the measured distribution of thedifferenceq=p2?p1between the three-momenta of the two par-ticlesp1andp2, andB(q)is the corresponding distribution formedby using pairs of particle. That means, experimentally one counts how many pairs with a certain q one has and normalizes that (more or less). What matters is the width of the resulting distribution which is what will yield the HBT radii (after fitting).

The expansion of the fireball decreases the HBT radii. It seems counterintuitive (at least to me) since as bigger the system one should have larger HBT radii right? However if you look at slide 33 of the lecture you will clearly see that collective motion implies smaller HBT radii and expansion implies collective motion.

Throughout the exercise sheet, a system of units such that the constants c and $\hbar$ equal 1 is used.

20. A toy model for the emission function

Consider a model with the emission function

$$S(\mathsf{x},\mathsf{p}) = \frac{NE_{\vec{p}}}{(2\pi R_s P_0)^3}\exp\left[-\frac{1}{2(1-s^2)}\left(\frac{\vec{r}^{\,2}}{R_0^2} - 2s\frac{\vec{r}\cdot\vec{p}}{R_0P_0} + \frac{\vec{p}^{\,2}}{P_0^2}\right)\right]\delta(t-t_{\mathrm{f.o.}}) \tag{1}$$

where N, R_0, P_0, $0 \le s < 1$, and $t_{\mathrm{f.o.}}$ are parameters while $R_s \equiv R_0\sqrt{1-s^2}$. On the left hand side, we have used the notations $\mathsf{x} \equiv (t,\vec{r})$, $\mathsf{p} \equiv (E_{\vec{p}},\vec{p})$.

Suggest an interpretation for the parameters. Compute the two-particle correlation function $\mathcal{C}(\vec{K},\vec{q})$ using the two formulas

$$\mathcal{C}(\vec{K},\vec{q}) = 1 + \frac{\left|\int S(\mathsf{x},\mathsf{K})\,e^{i\mathsf{q}\cdot\mathsf{x}}\,d^4\mathsf{x}\right|^2}{\int S\left(\mathsf{x},\mathsf{K}+\frac{\mathsf{q}}{2}\right)d^4\mathsf{x}\int S\left(\mathsf{x},\mathsf{K}-\frac{\mathsf{q}}{2}\right)d^4\mathsf{x}} \quad\text{and}\quad \mathcal{C}(\vec{K},\vec{q}) = 1 + \frac{\left|\int S(\mathsf{x},\mathsf{K})\,e^{i\mathsf{q}\cdot\mathsf{x}}\,d^4\mathsf{x}\right|^2}{\left[\int S(\mathsf{x},\mathsf{K})\,d^4\mathsf{x}\right]^2} \tag{2}$$

and compare the results.

Solution:

We see that the emission function has quite a few parameters.

R_o can be understood as a typical system size or source size and similarly P_o in momentum space. N might be a normalization factor and $t_{f.o.}$ is the freeze-out time which means that all particles are created at a single time point. s is a bit less obvious, we can say that s is a parameter that controls the amount of correlation between the spatial and momentum distributions.

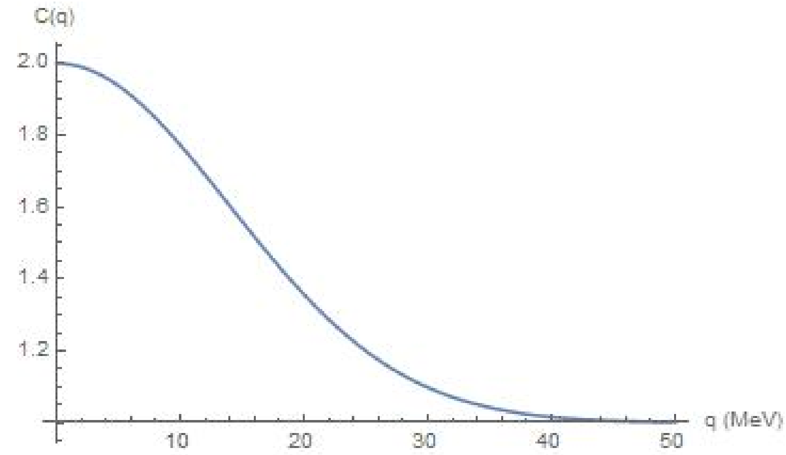

Figure 1: Correlation function (simplified one) for R_s =10fm as function of q.

Please check the Mathematica notebook for the following calculations. For the first two-particle correlation we find

$$\mathcal{C}(\vec{K},\vec{q}) = 1 + \frac{\left|\int S(\mathsf{x},\mathsf{K})\,\mathrm{e}^{\mathrm{i}\mathsf{q}\cdot\mathsf{x}}\,\mathrm{d}^4\mathsf{x}\right|^2}{\int S\left(\mathsf{x},\mathsf{K}+\frac{\mathsf{q}}{2}\right)\mathrm{d}^4\mathsf{x}\int S\left(\mathsf{x},\mathsf{K}-\frac{\mathsf{q}}{2}\right)\mathrm{d}^4\mathsf{x}} = 1 + e^{-q^2R_s^2(1-\frac{1}{(2R_sP_o)^2})}.$$

The second "version" of the correlation function is nothing else than a simplification of the first where the momentum pair difference is assumed to be smaller than the average i.e. $|\vec{K}| >> |\vec{q}|$. This allows us to simplify the denominator into an easier expression (this is a quite common and useful simplification, specially if you want to do analytical calculations).

$$\mathcal{C}(\vec{K},\vec{q}) = 1 + \frac{\left|\int S(\mathsf{x},\mathsf{K})\,\mathrm{e}^{\mathrm{i}\mathsf{q}\cdot\mathsf{x}}\,\mathrm{d}^4\mathsf{x}\right|^2}{\left[\int S(\mathsf{x},\mathsf{K})\,\mathrm{d}^4\mathsf{x}\right]^2} = 1 + e^{-q^2R_s^2}.$$

We can see that both expressions lead to similar results but in one case the width of the Gaussian is larger than in the other. Notice that, if R_s and/or P_o are large, then both expressions coincide.

We can see that the correlation function is maximum for zero momentum pair difference and it goes towards 1. This means that the correlations is "disappearing" for large q. Similar results would be found for the full correlation but then we need to know the values of s and P_o. Q: Why is $q^2R_s^2$ dimensionless?

21. Bose–Einstein statistics and pair correlation functions

Consider pions described by plane waves of the form $\psi = \mathcal{N}\,\mathrm{e}^{\mathrm{i}\mathsf{p}\cdot\mathsf{x}}$ where $\mathcal{N}$ is a “normalization” constant that plays no role in the following.

i. Write down (symmetrization!) a normalized wave function Ψ_{12} for a system of two pions with four-momenta p_1, p_2 and positions x_1, x_2. Express Ψ_{12} in terms of $\mathsf{K} \equiv \frac{1}{2}(\mathsf{p}_1 + \mathsf{p}_2)$ and $\mathsf{q} \equiv \mathsf{p}_1 - \mathsf{p}_2$.

Solution: In principle we would write the wave function as the product of the plane waves as

$$\Psi_{12} = N^2 e^{i(p_1\cdot x_1 + p_2\cdot x_2)}$$

but this wave function is supposed to represent a system of two identical pions which are bosons and therefore the wave function should be symmetric. The symmetrized wave function is then

$$\Psi_{12} = \frac{N^2}{\sqrt{2}}(e^{i(p_1\cdot x_1 + p_2\cdot x_2)} + e^{i(p_1\cdot x_2 + p_2\cdot x_1)})$$

which is now symmetric. Question: Why the factor $\frac{1}{\sqrt{2}}$?

We can rewrite it simply by using that

$$\vec{p_1} = \vec{K} + \vec{q}/2 \qquad \vec{p_2} = \vec{K} - \vec{q}/2$$

which yields

$$\Psi_{12} = \frac{N^2}{\sqrt{2}} e^{iK\cdot(x_1+x_2)}(e^{i\frac{q}{2}\cdot(x_1-x_2)} + e^{-i\frac{q}{2}\cdot(x_1-x_2)})$$

ii. Defining single- and pair-momentum probabilities by

$$\mathcal{P}_1(\mathsf{p}) \equiv \int \rho(\mathsf{x})|\psi|^2\,\mathrm{d}^4\mathsf{x} \quad , \quad \mathcal{P}_2(\mathsf{p}_1,\mathsf{p}_2) \equiv \int \rho(\mathsf{x}_1)\rho(\mathsf{x}_2)|\psi_{12}|^2\,\mathrm{d}^4\mathsf{x}_1\,\mathrm{d}^4\mathsf{x}_2, \tag{3}$$

compute the pair correlation function $\mathcal{C}(\vec{K},\vec{q}) \equiv \mathcal{P}_2(\mathsf{p}_1,\mathsf{p}_2)/\big[\mathcal{P}_1(\mathsf{p}_1)\mathcal{P}_1(\mathsf{p}_2)\big]$.

Solution:

Following the form of the single-momentum probability we find

$$\mathcal{P}_1(\mathsf{p}) \equiv \int \rho(\mathsf{x})|\psi|^2\,\mathrm{d}^4\mathsf{x} = \int \rho(\mathsf{x})|Ne^{ip\cdot x}|^2 d^4x = N^2 \int \rho(\mathsf{x}) d^4x.$$

We will compute now the pair-momentum probability, but first we compute

$$|\Psi_{12}|^2 = \frac{N^2}{2}(2 + (e^{iq\cdot(x_1-x_2)} + e^{-iq\cdot(x_1-x_2)}))$$

The pair-momentum probability reads as

$$\mathcal{P}_2(\mathsf{p}_1,\mathsf{p}_2) \equiv \int \rho(\mathsf{x}_1)\rho(\mathsf{x}_2)|\Psi_{12}|^2\,\mathrm{d}^4\mathsf{x}_1\,\mathrm{d}^4\mathsf{x}_2 =$$

$$= N^4 \int \rho(\mathsf{x}_1)\rho(\mathsf{x}_2)\,\mathrm{d}^4\mathsf{x}_1\,\mathrm{d}^4\mathsf{x}_2) + \frac{N^4}{2}\int \rho(\mathsf{x}_1)\rho(\mathsf{x}_2)(e^{iq\cdot(x_1-x_2)} + e^{-iq\cdot(x_1-x_2)})\,\mathrm{d}^4\mathsf{x}_1\,\mathrm{d}^4\mathsf{x}_2).$$

The first integral is just $\mathcal{P}_1(\mathsf{p})^2$ since the integrals can be separated into one that only depends on x_1 and the other x_2. The second integral can be written as

$$\frac{N^4}{2}\int \rho(\mathsf{x}_1)\rho(\mathsf{x}_2)(e^{iq\cdot(x_1-x_2)} + e^{-iq\cdot(x_1-x_2)})\,\mathrm{d}^4\mathsf{x}_1\,\mathrm{d}^4\mathsf{x}_2) = N^4 |\int \rho(\mathsf{x}) e^{-i\vec{q}\cdot\vec{x}} d^4x|^2$$

where you can again separate it into the product of two integrals.

All together, the pair correlation function is

$$\mathcal{C}(\vec{K},\vec{q}) \equiv \mathcal{P}_2(\mathsf{p}_1,\mathsf{p}_2)/\big[\mathcal{P}_1(\mathsf{p}_1)\mathcal{P}_1(\mathsf{p}_2)\big] = 1 + \frac{|\int \rho(\mathsf{x})e^{-i\vec{q}\cdot\vec{x}}d^4x|^2}{\left(\int \rho(\mathsf{x})d^4x\right)^2}.$$

If we want to go further we need to specify $\rho(x)$ which we are not gonna do. However you can already see that for small values of q (or more precisely for $q = 0$) the correlations functions is equal to 2. That means that the pions are fully correlated.