# Solutions to tutorial sheet 1

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Please feel absolutely free to send me emails with questions, doubts, complains, etc. My purpose is to help you understand as much as possible about Heavy Ion Physics and have fun in the meantime.

We will use a system of units (natural units) in which $\hbar = c = k_B = 1$. This implies, for example, that the mass of a particle will be given in units of energy and the time will have units of length (it sound weird the first time).

**Discussion topic:** In which sense are quarks "free" in a quark-gluon plasma? How does this relate to QCD?

## 1. Collision kinematics

In a heavy-ion collision at a collider, the nuclei that collide move (along the $z$-axis) with velocities $+v$ and $-v$ with respect to the "laboratory frame" in which the detectors are at rest. For simplicity, a nucleus will be assimilated to a sphere of radius $R$.

**i.** What is the size ("diameter") of one of the nuclei along the $z$-axis from the point of view of an observer at rest in the laboratory?
When the nuclei collide, they fly through each other without being decelerated — to some extent, they are "transparent". How long does it take in the laboratory frame for the two nuclei to cross each other?

**Solution:** Along the transverse (x and y) directions there is no change in size since the nuclei travel along the longitudinal (z) direction. In the longitudinal direction the gamma factor will be

$$\gamma = \frac{1}{\sqrt{1-v^2}}. \tag{1}$$

From spatial relativity we know that an object of initial size D that moves with a gamma factor $\gamma$ will get contracted if observed in the laboratory frame by

$$D_{lab} = \frac{D}{\gamma} \tag{2}$$

where D is, in our case, the diameter of a nucleus.

We assume now that the two nuclei cross each other without being decelerated (which is actually mostly true). In such case, an observer in the laboratory frame will see that at t=0 the two nuclei are in front of each other but no collision has happened yet. At a time $t = D_{lab}$ the two nuclei will have crossed each other and will be flying away.

**ii.** At the Large Hadron Collider, lead nuclei — (rest) mass $m_{\mathrm{Pb}} \simeq 194$ GeV, "radius" $R_{\mathrm{Pb}} \simeq 6.6$ fm — are accelerated to an energy $E \approx 522$ TeV each. How long does it take for the two Pb nuclei to cross each other? You may express your answer either in fm/$c$ or in yoctoseconds (1 ys $= 10^{-24}$ s). Why are the colliding nuclei often referred to as "pancakes"?

**Solution:** First of all we want to compute the gamma factor. This can be easily done by using that

$$E = m\gamma \tag{3}$$

which leads to

$$\gamma = \frac{522TeV}{194GeV} \approx 2700$$

With that we can compute the "new" radii of lead nuclei which becomes

$$R_{lab} = 6.6fm/2700 \approx 0.0025fm$$

This means that 400 lead nuclei in line would occupy 1fm in the longitudinal direction.

The amount of time that it takes for two nuclei to cross each other will be

$$t = D_{lab} = 0.005fm$$

which is effectively zero.

This is why the nuclei are often called as pancakes.

As extra information, the time will get dilated by also a factor 2700. Therefore, you can try to imagine that some processes (inside the nuclei) that live for very short times will now live much longer.

## 2. A useful set of coordinates

To describe the system created in a heavy ion collision, especially at high energies, it is customary to replace the time $t$ and the "longitudinal" spatial coordinate $z$ along the flight direction of the colliding nuclei by other, equivalent coordinates: the "proper time" $\tau$ and "spatial rapidity"[1] $\varsigma$

$$\tau \equiv \sqrt{t^2 - z^2} \quad , \quad \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \text{ where } |z| < t, \tag{4}$$

in a system of units in which $c = 1$. Implicit in this definition is that the nuclei met at $z = 0$ at time $t = 0$.

**i.** Check that the relations defining $\tau$ and $\varsigma$ can be inverted, yielding[2]

$$t = \tau \cosh \varsigma \quad , \quad z = \tau \sinh \varsigma. \tag{5}$$

(Hint: $\frac{1}{2} \log \frac{1+u}{1-u}$ has a simpler expression in terms of hyperbolic functions.)

**Soulution:** As suggested, we compute first the following (sometines the spatial rapidity is written as $\eta_s$)

$$cosh\varsigma = cosh\left(ln\left(\frac{\sqrt{t+z}}{\sqrt{t-z}}\right)\right) = \frac{t}{\sqrt{t^2 - z^2}} = \frac{t}{\tau}$$

We have showed the first relation. The second follows similarly by computing the hyperbolic sine instead of the cosine.

**ii.** Draw on a space-time diagram — with $t$ on the vertical axis and $z$ on the horizontal axis — the lines of constant $\tau$ and those of constant $\varsigma$.

**Soulution:** Curves with constant proper time will be parabolas. Meanwhile curves with constant spatial rapidity imply that the fraction $\frac{t+z}{t-z}$ is a constant, which is the equation of a line.

[1] This is the LaTeX character `\varsigma`

[2] Note the analogy with polar coordinates $(r, \theta)$!

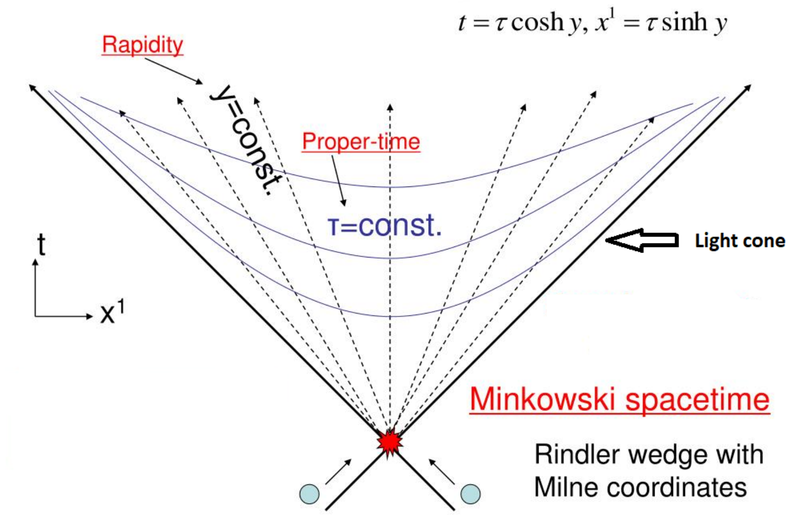


Figure 1: Different lines with constant proper time or spatial rapidity. (what is called $y$ corresponds to the spatial rapidity).

**iii.** What is the motion of an observer $\mathcal{O}_\varsigma$ at a constant finite $\varsigma$ with respect to (hereafter abbreviated: w.r.t.) an observer $\mathcal{O}_0$ at rest at $z = 0$? In particular, how is $\varsigma$ related to the velocity of $\mathcal{O}_\varsigma$ w.r.t. $\mathcal{O}_0$?

Consider now a third observer $\mathcal{O}_{\varsigma'}$ at spatial rapidity $\varsigma'$ w.r.t. $\mathcal{O}_0$. Can you guess (no calculation!) what is the spatial rapidity of $\mathcal{O}_{\varsigma'}$ w.r.t. $\mathcal{O}_\varsigma$ — that is, the spatial rapidity of $\mathcal{O}_{\varsigma'}$ you would obtain by using Lorentz coordinates comoving with $\mathcal{O}_\varsigma$, and thus properly boosted along the $z$-direction w.r.t. the original $(t, z)$-coordinates of $\mathcal{O}_0$?

**Soulution:** An observer at z=0 ($\mathcal{O}_0$ ). Another observer is situated at a fixed spatial rapidity, that means that $\frac{t+z}{t-z}$ of that observer is fixed. The proper time is boost invariant (you can easily show that). However, the spatial rapidity is not boost invariant.

Under a Lorentz boost along the z-axis it can be written that

$$t' = \gamma(t - zv_z) \qquad z' = \gamma(z - tv_z) \tag{6}$$

which are the usual Lorentz transformations (with c=1). I will let you proof that the proper time is invariant.

The spatial rapidity observed by the observer at z=0 will be

$$\varsigma' = \frac{1}{2}ln(\frac{t'+z'}{t'-z'}) = ... = \frac{1}{2}ln(\frac{t+z}{t-z}) + \frac{1}{2}ln(\frac{1+v_z}{1-v_z}) = \varsigma + y \tag{7}$$

where $y = \frac{1}{2}ln(\frac{1+v_z}{1-v_z})$ is called rapidity and $v_z = tanh(y)$. Therefore, the observer at $\mathcal{O}_\varsigma$ is moving with a velocity that is $v_z = tanh(\varsigma)$.

We have showed the spatial rapidity is additive under a Lorentz boost along z. Therefore, if there is a third observer at a spatial rapidity $\varsigma'$, the observer at $\varsigma$ will see this third observer with a spatial rapidity that corresponds to

$$\varsigma_{relative} = \varsigma' - \varsigma$$

.

This also applies to the rapidity (y). For example, if you have a problem that you can solve in the center of mass frame then you can boost back to the laboratory frame by using that

$$y_{lab} = y_{CM} + y$$

.

## 3. A problem of collider physics

This exercise should help you appreciate one of the difficulties faced by the people tuning the behavior of a collider to ensure that two beams of particles circulating in opposite directions meet at the desired point, inside a detector.

Put your forearms parallel to each other, about 50 cm away from each other. Turn your hands inwards, and point your left and right forefingers — that's their job! — towards each other. Close your eyes (after reading the sentence till the end!) and move your hands inwards: how many attempts will you need until the tips of your forefingers meet "head on"?

At the LHC, the proton beams at the interaction points have a sub-millimeter transverse size...

**Comment:** At the LHC the beam size is of the order of the micrometer. That means that two beams of particles circulating in opposite directions have to meet within less than one micrometer difference.