High energy hadronic interactions in QCD and applications to heavy ion collisions

V – Calculating observables in the CGC

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General outline

- Lecture I : Introduction and phenomenology
- Lecture II : Lessons from Deep Inelastic Scattering
- Lecture III : QCD on the light-cone
- Lecture IV : Saturation and the Color Glass Condensate
- Lecture V : Calculating observables in the CGC

- Field theory coupled to time-dependent sources
- Generating function for the probabilities
- Average particle multiplicity
- Numerical methods for nucleus-nucleus collisions
  - Gluon production
  - Quark production

Introduction

- In the Color Glass Condensate framework, hadronic projectiles are described by a bunch of color sources flying at the speed of light :

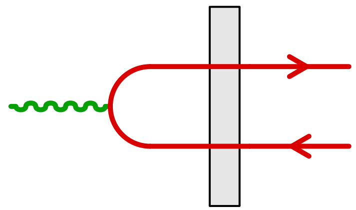
$$J^\mu = \delta^{\mu +}\delta(x^-)\rho(\vec{\boldsymbol{x}}_\perp)$$

- In the previous lecture, we have studied properties of the statistical distribution $W_Y[\rho]$ of these sources, in particular its evolution under changes of the rapidity Y
- Now, we focus on another aspect of the problem:

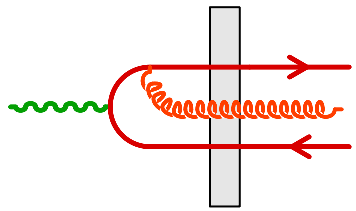
  given the sources, how do we calculate observable quantities for hadronic collisions ?

Introduction

- The case of the interaction between a proton or nucleus described by the CGC and an "elementary" probe is fairly simple. The archetype of this situation is Deep Inelastic Scattering (the elementary probe being a virtual photon)
- At lowest order, one simply considers the interaction with the proton of a $Q\overline{Q}$ fluctuation of the virtual photon :



- More complicated Fock states should in principle be considered as well at higher orders :



Introduction

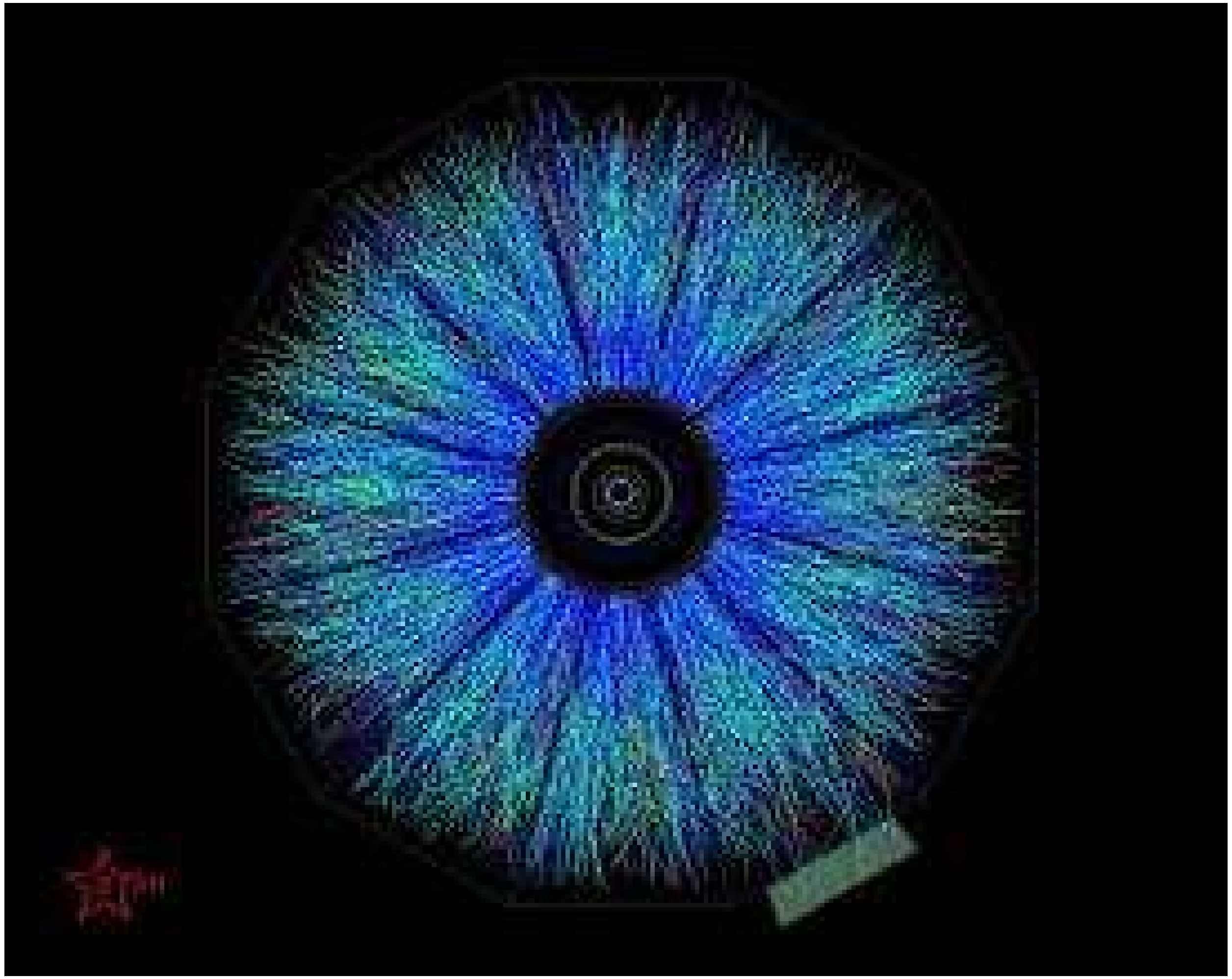
- Interactions between two hadrons described by the CGC are treated by coupling the fields to a source which is the sum of two terms :

$$J^\mu = \delta^{\mu +}\delta(x^-)\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu -}\delta(x^+)\rho_2(\vec{\boldsymbol{x}}_\perp)$$

- If one of the sources is weak (i.e. the corresponding hadron has a parton density much smaller than the other hadron), the problem is again rather easy to study
  - In this case, one computes the transition amplitudes at lowest order in the weak source, which is much easier than keeping both of them to all orders
- In this lecture, I will consider only the collision of two very dense hadrons, for which no such approximation is possible

Introduction

- In short, our goal is to say something useful about this...



Introduction

- From now on, we assume that $j = j_1 + j_2$, with j_1 and j_2 of comparable strengths

- The sources can be as strong as $1/g$ in the saturated regime: ▷ corrections in $(gj)^n$ must be summed to all orders, which makes the evaluation of physical quantities very complicated – even at "leading order"

- In fact, very few physical quantities are calculable by simple methods when this resummation is necessary, and it is important to know which ones...

- To avoid encumbering the discussion with unessential (for now) details, we first consider a scalar field theory with a ϕ^3 coupling, coupled to a source $j(x)$:

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 + j\phi$$

Counting the powers of g

- Consider a diagram with :
  - n_E external lines
  - n_I internal lines
  - n_V vertices
  - n_J sources
  - n_L independent loops
- These numbers are related by :

$$3n_V + n_J = n_E + 2n_I$$
$$n_L = n_I - n_J + 1$$

- Therefore, the order of the diagram in g and j is :

$$g^{n_V} j^{n_J} = g^{n_E+2(n_L-1)}(gj)^{n_J}$$

- After resummation of all the powers of gj, the order of a diagram depends only on its number of loops and external legs

Counting the powers of h

- To calculate the order of a diagram in $\hbar$, remember that the evolution operator is in fact $\exp(S/\hbar)$, with :

$$\frac{S}{\hbar} = \int d^4x \left[-\frac{1}{2}\phi \frac{\square + m^2}{\hbar}\phi - \frac{g}{3!\hbar}\phi^3 + \frac{j}{\hbar}\phi \right]$$

- Therefore, the powers of $\hbar$ come as follows :
  - a power of $\hbar$ for each propagator
  - a power of $1/\hbar$ for each vertex
  - a power of $1/\hbar$ for each source
- The order in $\hbar$ of a diagram is :

$$\hbar^{n_E + n_I - n_V - n_J} = \hbar^{n_E + n_L - 1}$$

- When one resums the corrections in $(gj)^n$, all the included terms have the same order in $\hbar$

Asymptotic free fields

- From the Heisenberg field operator $\phi(x)$, one can define two free fields $\phi_{\rm in}(x)$ and $\phi_{\rm out}(x)$, which coincide with $\phi(x)$ respectively at $t=-\infty$ and $t=+\infty$:

$$\begin{aligned}\phi(x) &= U(-\infty,x^0)\phi_{\rm in}(x)U(x^0,-\infty)\\ \phi(x) &= U(+\infty,x^0)\phi_{\rm out}(x)U(x^0,+\infty)\end{aligned}$$

with $\quad U(t_2,t_1)\equiv \mathcal{P}\exp i\int_{t_1}^{t_2} d^4x\,\mathcal{L}_{\rm int}(\phi_{\rm in/out}(x))$

- These free fields have a simple Fourier decomposition :

$$\phi_{\rm in/out}(x)=\int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_k}\left[a_{\rm in/out}(\vec{\boldsymbol{k}})e^{-ik\cdot x}+a^\dagger_{\rm in/out}(\vec{\boldsymbol{k}})e^{+ik\cdot x}\right]$$

- The in and out creation/annihilation operators are related by :

$$\begin{aligned}a^\dagger_{\rm out}(\vec{\boldsymbol{k}}) &= U(-\infty,+\infty)a^\dagger_{\rm in}(\vec{\boldsymbol{k}})U(+\infty,-\infty)\\ a_{\rm out}(\vec{\boldsymbol{k}}) &= U(-\infty,+\infty)a_{\rm in}(\vec{\boldsymbol{k}})U(+\infty,-\infty)\end{aligned}$$

In and out states, S-matrix

- To the in and out creation/annihilation operators, one associates in and out states :

vacuum state : $|0_{\rm in}\rangle, |0_{\rm out}\rangle$

1-particle states : $|\vec{\boldsymbol p}_{\rm in}\rangle = a^\dagger_{\rm in}(\vec{\boldsymbol p})|0_{\rm in}\rangle$

$|\vec{\boldsymbol p}_{\rm out}\rangle = a^\dagger_{\rm out}(\vec{\boldsymbol p})|0_{\rm out}\rangle$

- From the relationship between $a^\dagger_{\rm in}$ and $a^\dagger_{\rm out}$, one gets :

$$|\alpha_{\rm in}\rangle = U(+\infty,-\infty)|\alpha_{\rm out}\rangle \qquad \text{for any state } \alpha$$

- $U(+\infty,-\infty)$ is the S-matrix. Indeed :

$$S_{\beta\alpha} \equiv \big\langle\beta_{\rm out}\big|\alpha_{\rm in}\big\rangle = \big\langle\beta_{\rm in}\big|S\big|\alpha_{\rm in}\big\rangle = \big\langle\beta_{\rm in}\big|U(+\infty,-\infty)\big|\alpha_{\rm in}\big\rangle$$

  - Even if the fields are not self-interacting, the S-matrix differs from 1 because of the source j : particles can scatter off the external field. If j is time-dependent, it can even create particles

Reduction formulas

- In order to express transition amplitudes in terms of field operators, we need the following relations :

$$a_{\rm in/out}(\boldsymbol{\vec{k}}) = i\int d^3\boldsymbol{\vec{x}}\; e^{ik\cdot x}\;\Big[\partial_0 - iE_k\Big]\;\phi_{\rm in/out}(x)$$

- Production of a single particle :

$$\big<\boldsymbol{p}_{\rm out}\big|0_{\rm in}\big> = \frac{1}{Z^{1/2}}\int d^4x\; e^{ip\cdot x}\;(\square_x + m^2)\big<0_{\rm out}\big|\phi(x)\big|0_{\rm in}\big>$$

- Production of a two particles :

$$\big<\boldsymbol{\vec{p}}\boldsymbol{\vec{q}}_{\rm out}\big|0_{\rm in}\big> = \frac{1}{Z}\int d^4x\,d^4y\; e^{iq\cdot y}e^{ip\cdot x}$$
$$\times(\square_x + m^2)(\square_y + m^2)\big<0_{\rm out}\big|T\phi(x)\phi(y)\big|0_{\rm in}\big>$$

Perturbative expansion

- In order to calculate $\big\langle 0_{\rm out}\big|T\phi(x_1)\cdots\phi(x_n)\big|0_{\rm in}\big\rangle$ in perturbation theory, we must write :

$$\begin{aligned}
&\big\langle 0_{\rm out}\big|T\phi(x_1)\cdots\phi(x_n)\big|0_{\rm in}\big\rangle = \\
&\quad = \big\langle 0_{\rm in}\big|U(+\infty,-\infty)TU(-\infty,x_1^0)\phi_{\rm in}(x_1)U(x_1^0,x_2^0)\cdots \\
&\qquad\qquad \cdots U(x_{n-1}^0,x_n^0)\phi_{\rm in}(x_n)U(x_n^0,-\infty)\big|0_{\rm in}\big\rangle \\
&\quad = \big\langle 0_{\rm in}\big|T\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)e^{i\int_{-\infty}^{+\infty}d^4x\,\mathcal{L}_{\rm int}(\phi_{\rm in})}\big|0_{\rm in}\big\rangle
\end{aligned}$$

- Now that everything has been rewritten in terms of the free field $\phi_{\rm in}$, it is just a matter of expanding the exponential to the desired order
- Note that this expansion generates vacuum-vacuum diagrams, whose sum appears as a multiplicative prefactor.

  If $j=0$, the sum of the vacuum-vacuum diagrams, $\big\langle 0_{\rm out}\big|0_{\rm in}\big\rangle$, is a pure phase and can be disregarded from squared amplitudes. This is not the case here

Vacuum-vacuum diagrams

- A source $j(x)$ that describes a single projectile does not produce particles. Indeed, it is static in the rest-frame of this projectile, and therefore can only have space-like modes
- A source $j(x) \equiv j_1(x) + j_2(x)$ describing two projectiles moving in opposite directions can produce particles
- If the transition amplitudes $\langle \boldsymbol{\vec{p}} \cdots_{\rm out} | 0_{\rm in} \rangle$ are non-zero, then the vacuum-to-vacuum transition amplitude $\langle 0_{\rm out} | 0_{\rm in} \rangle$ is non trivial. Indeed,

$$\sum_\alpha \left| \langle \alpha_{\rm out} | 0_{\rm in} \rangle \right|^2 = 1 \qquad \text{(unitarity)}$$

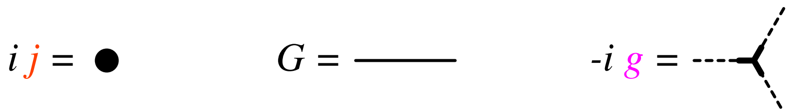
implies $\left| \langle 0_{\rm out} | 0_{\rm in} \rangle \right|^2 < 1$ if at least one of the $\langle \boldsymbol{\vec{p}} \cdots_{\rm out} | 0_{\rm in} \rangle$ is non-zero
- On the contrary, when $j(x)$ cannot produce particles, then the vacuum-to-vacuum amplitude is a pure phase

Vacuum-vacuum diagrams

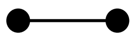
- The sum of all the vacuum-vacuum diagrams in $\big\langle 0_{\text{out}} \big| 0_{\text{in}} \big\rangle$ is equal to the exponential of the sum of the connected ones

$$\big\langle 0_{\text{out}} \big| 0_{\text{in}} \big\rangle = e^{i \sum_{\text{conn}} V}$$

- Let us denote :



- The perturbative expansion of $i \sum_{\text{conn}} V$ starts with :



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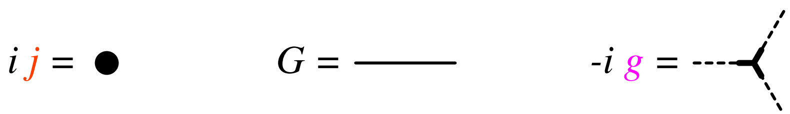

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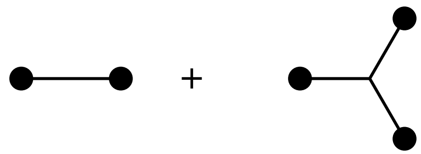

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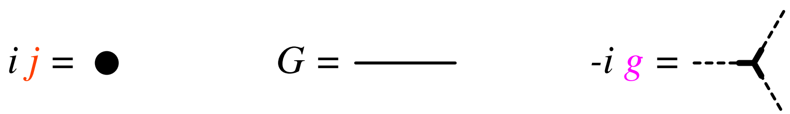

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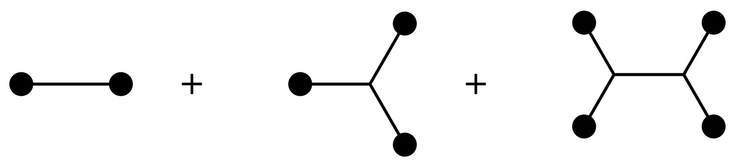

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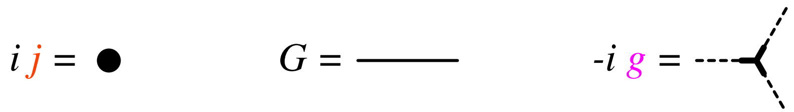

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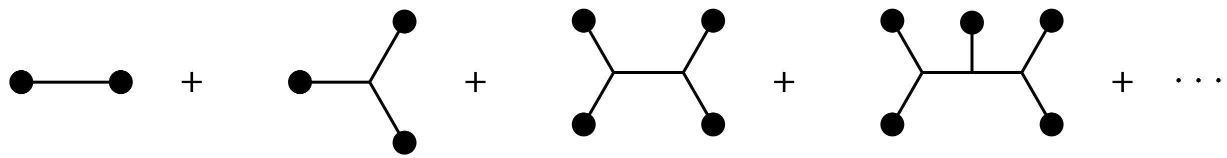

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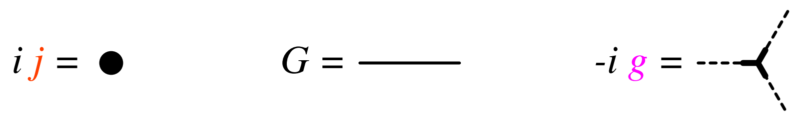

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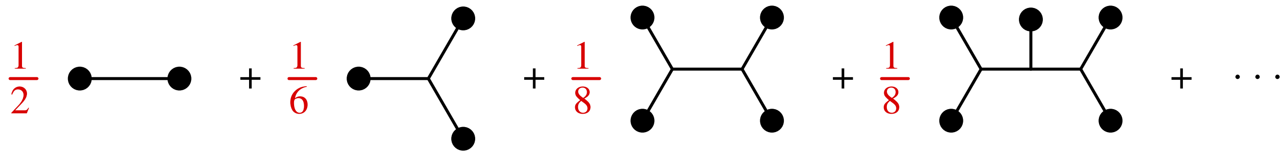

Note : each graph Γ comes with a symmetry factor $1/S_\Gamma$, where S_Γ is the order of its symmetry group

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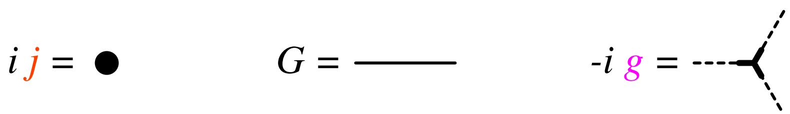

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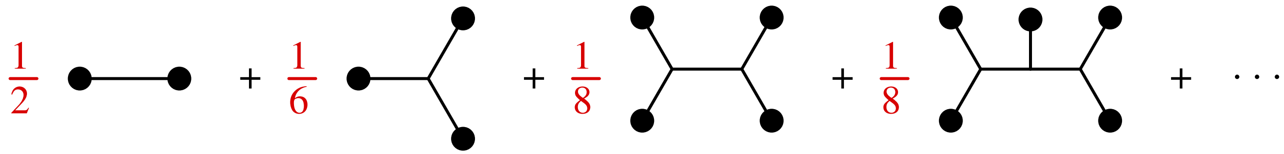


Note : each graph Γ comes with a symmetry factor $1/S_\Gamma$, where S_Γ is the order of its symmetry group

- Note : $\big\langle 0_{\rm out}\big|0_{\rm in}\big\rangle$ can be seen as a generating functional :

$$\big\langle 0_{\rm out}\big|T\phi(x_1)\cdots\phi(x_n)\big|0_{\rm in}\big\rangle = \frac{\delta}{i\delta j(x_1)}\cdots\frac{\delta}{i\delta j(x_n)}\;\big\langle 0_{\rm out}\big|0_{\rm in}\big\rangle$$

Definitions

- The probability of producing exactly n particles in the collision of the two hadrons (in addition to the fragments of the incoming hadrons) is given by :

$$P_n = \frac{1}{n!}\int \frac{d^3\vec{\boldsymbol{p}}_1}{(2\pi)^3 2E_1}\cdots\frac{d^3\vec{\boldsymbol{p}}_n}{(2\pi)^3 2E_n}\left|\left\langle \vec{\boldsymbol{p}}_1\cdots\vec{\boldsymbol{p}}_{n\,\text{out}}\middle|0_{\text{in}}\right\rangle\right|^2$$

- One can define a generating function : $F(x) \equiv \sum_{n=0}^{+\infty} P_n\, e^{nx}$
- The sum of all the P_n must be 1, hence $F(0) = 1$
- From $F(x)$, it is very easy to obtain the moments of the distribution :

$$\left\langle n^p \right\rangle = \sum_{n=0}^{+\infty} n^p\, P_n = F^{(p)}(0)$$

- Note : connected moments, e.g. $\left\langle n^2 \right\rangle - \left\langle n \right\rangle^2$, are obtained by differentiating $\ln(F(x))$ instead of $F(x)$

Probability Pn

- Denote $\exp(iV[j])$ the sum of all vacuum-vacuum diagrams
- The reduction formula can be written as :

$$\langle \vec{\boldsymbol{p}}_1 \cdots \vec{\boldsymbol{p}}_{n\,\mathrm{out}} | 0_{\mathrm{in}} \rangle = \frac{1}{Z^{n/2}} \int \left[\prod_{i=1}^{n} d^4x_i \; e^{ip_i\cdot x_i} \; (\Box_i + m^2) \frac{\delta}{i\delta j(x_i)} \right] e^{iV[j]}$$

and we have $\quad P_n = \frac{1}{n!}\, \mathcal{D}^n[j_+, j_-]\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$

with

$$\mathcal{D}[j_+, j_-] \equiv \frac{1}{Z} \int_{x,y} G^0_{+-}(x,y)\; (\Box_x + m^2)(\Box_y + m^2)\; \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

$$G^0_{+-}(x,y) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_p}\; e^{ip\cdot(x-y)}$$

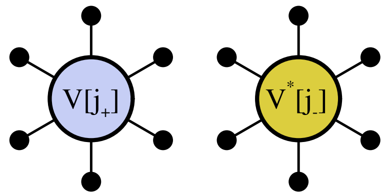
- Therefore, we have :

$$F(x) = e^{e^x \mathcal{D}[j_+,j_-]}\;\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

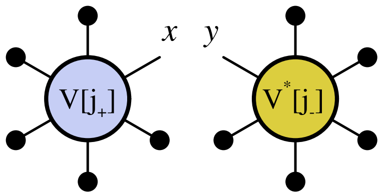
Action of D[j+,j-]

- Action of $\mathcal{D}[j_+, j_-]$:

$$\int_{x,y} G^0_{+-}(x,y)\,(\square_x + m^2)(\square_y + m^2)\,\frac{\delta}{\delta j_+(x)}\,\frac{\delta}{\delta j_-(y)}$$



$$= \int_{x,y} G^0_{+-}(x,y)\,(\square_x + m^2)(\square_y + m^2)$$



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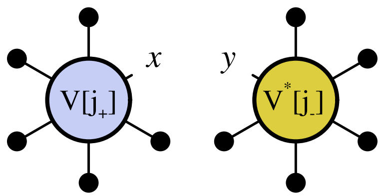

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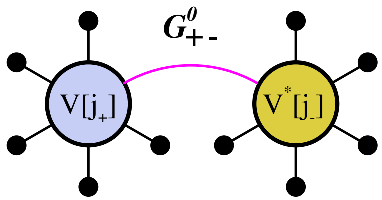


Cutting rules

- In order to interpret $F(x)$ in terms of diagrams, we need to discuss the "cutting rules" that give the imaginary part of a diagram
- Decompose the free time ordered propagator, G^0_{++}, as :

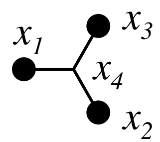
$$G^0_{++}(x,y) = \theta(x^0 - y^0)G^0_{-+}(x,y) + \theta(y^0 - x^0)G^0_{+-}(x,y)$$

- Define also :

$$G^0_{--}(x,y) \equiv \theta(x^0 - y^0)G^0_{+-}(x,y) + \theta(y^0 - x^0)G^0_{-+}(x,y)$$

- Consider a diagram in $i\sum_{\text{conn}} V$, before performing the space-time integrations : $iV(x_1 \cdots x_n)$. The x_i are the locations of the sources j and vertices g. For instance :

$$iV(x_1, x_2, x_3, x_4) \quad =$$



Cutting rules

- The diagrams iV are made only of the propagator G^0_{++}
- For each diagram $iV(x_1 \cdots x_n)$, construct 2^n diagrams $iV_{\epsilon_1 \cdots \epsilon_n}(x_1 \cdots x_n)$ where ϵ_i is a sign attached to the vertex i :
  - Connect a vertex of type ϵ to a vertex of type ϵ' by $G^0_{\epsilon\epsilon'}$
  - For vertices of type $-$, substitute $i\,g \to -i\,g$, $i\,j \to -i\,j$
- Largest time equation : if x_i^0 is the largest time in the diagram :

$$iV_{\cdots\epsilon_i\cdots}(x_1 \cdots x_n) + iV_{\cdots -\epsilon_i \cdots}(x_1 \cdots x_n) = 0$$

(the indices hidden in the dots are the same for both terms)

- Therefore, the sum of all the $iV_{\epsilon_1 \cdots \epsilon_n}$ is zero :

$$\sum_{\{\epsilon_i = \pm\}} iV_{\epsilon_1 \cdots \epsilon_n}(x_1 \cdots x_n) = 0$$

(group the terms in pairs, and use the previous result)

Cutting rules

- In momentum space, the propagators $G^0_{\epsilon\epsilon'}$ read :

$$G^0_{++}(p) = \frac{i}{p^2 - m^2 + i\epsilon} \qquad G^0_{--}(p) = \left[G^0_{++}(p)\right]^*$$

$$G^0_{-+}(p) = 2\pi\theta(p^0)\delta(p^2 - m^2) \qquad G^0_{+-}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

- Therefore, $iV_{++\cdots+}$ is the original diagram and $iV_{--\cdots-}$ is its complex conjugate

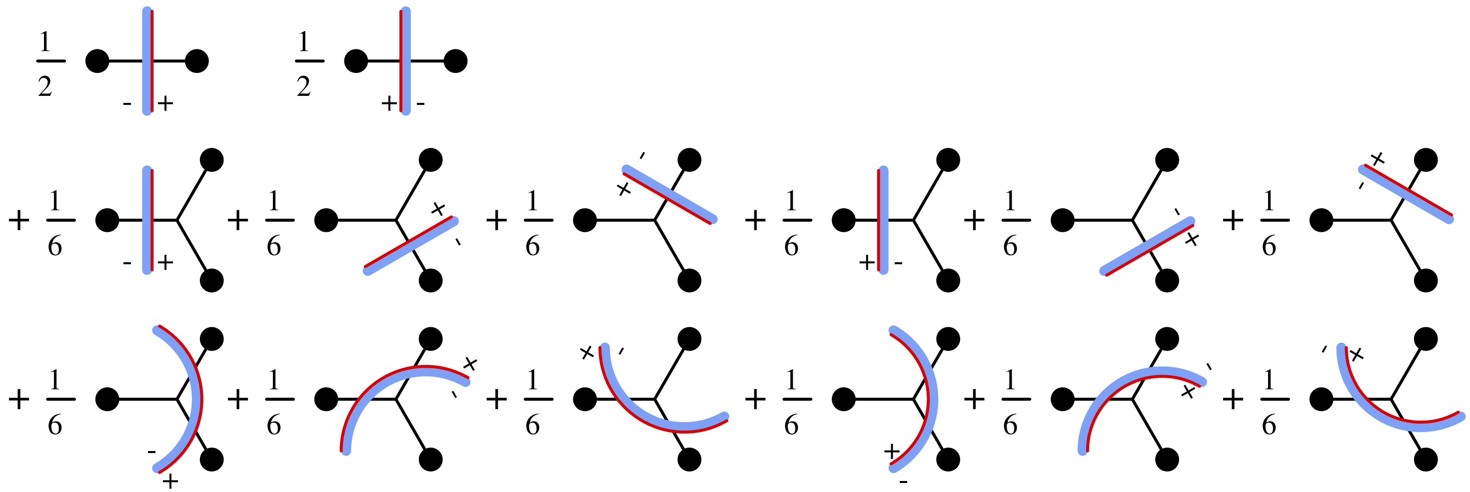
- By isolating these two terms from the sum over the 2^n terms, we get :

$$2\,\mathrm{Im}\,V = \sum_{\{\epsilon_i=\pm\}'} iV_{\epsilon_1\cdots\epsilon_n}$$

where the prime indicates that the sum over the ϵ_i's does not have the $++\cdots+$ and $--\cdots-$ terms

Cutting rules

- For each term in $\sum_{\{\epsilon_i=\pm\}'} iV_{\epsilon_1\cdots\epsilon_n}$, draw a line ("cut") separating the $+$ from the $-$ vertices
- The simplest terms in $2\,\mathrm{Im}\sum_{\mathrm{conn}} V$ are given by :



- Cuts through vacuum-vacuum diagrams are non-zero because of the coupling to the source j
- A cut going through r propagators will be called a r-particle cut

Interpretation of F(x)

- The generating function has the following interpretation :

  $F(x)$ is the sum of all the cut vacuum-vacuum diagrams (including the terms with only $+$ or only $-$), in which each term is weighted by a power of e^x equal to the number of particles on the cut

- Note : this implies automatically that $F(0) = 1$ (from the largest time equation)

- Let us denote b_r/g^2 the sum of all the r-particle cut connected vacuum-vacuum diagrams. Then, we have :

$$\ln(F(x)) = \sum_{r=1}^{+\infty} \frac{b_r}{g^2} \left(e^{rx} - 1\right)$$

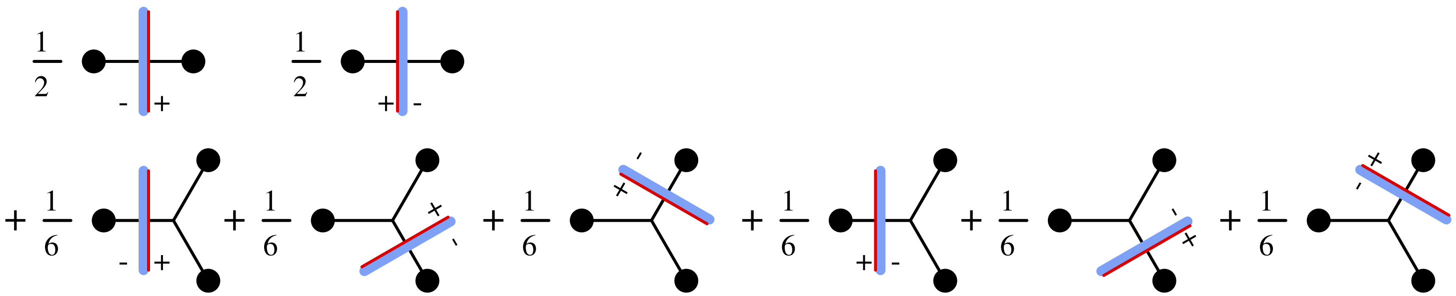
- This leads to the following formula for the connected moment of order p :

$$\left\langle n^p \right\rangle_{\text{conn}} = \frac{d^p}{dx^p} \ln(F(x))\Big|_{x=0} = \sum_{r=1}^{+\infty} r^p \frac{b_r}{g^2}$$

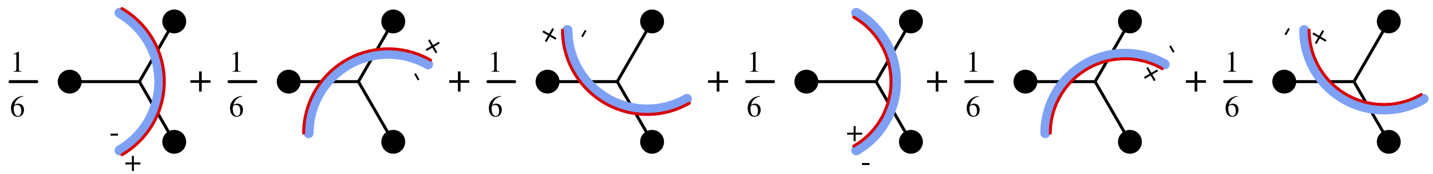
Interpretation of F(x)

- Lowest order diagrams in b_1/g^2, b_2/g^2, b_3/g^2 :

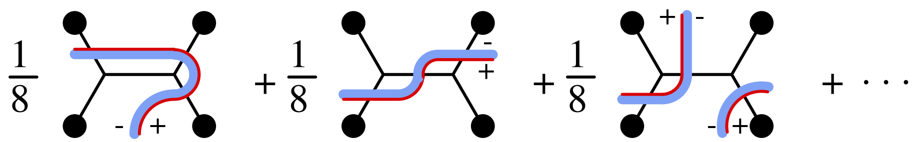
$$\frac{b_1}{g^2} =$$



$$\frac{b_2}{g^2} =$$



$$\frac{b_3}{g^2} =$$



Multiplicity distribution

- From the generating function, one obtains the following formula for the probability of producing n particles :

$$P_n = e^{-\sum_r b_r/g^2} \sum_{p=1}^{n} \frac{1}{p!} \sum_{\alpha_1+\cdots+\alpha_p=n} \frac{b_{\alpha_1}\cdots b_{\alpha_p}}{g^{2p}}$$

  - Note : in this formula, p is the number of disconnected subdiagrams producing the n particles, and α_i is the number of particles produced in the i-th subdiagram

- This is not a Poisson distribution

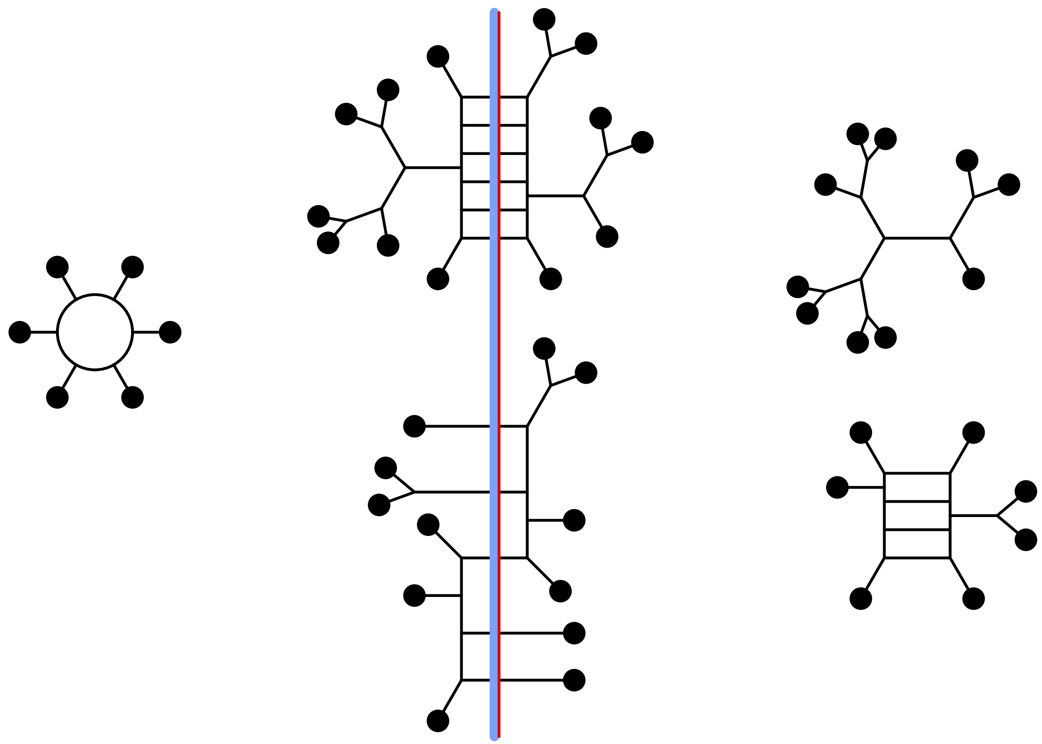
- In order to have a Poisson distribution, we would need :

$$b_r = 0 \qquad \text{for} \quad r \geq 2$$

i.e. all the particles produced in separate subdiagrams (if a subdiagram can produce more than one particle, this introduces correlations)

Multiplicity distribution

- Example : contribution to P_{11} with a bit of b_5 and b_6 :



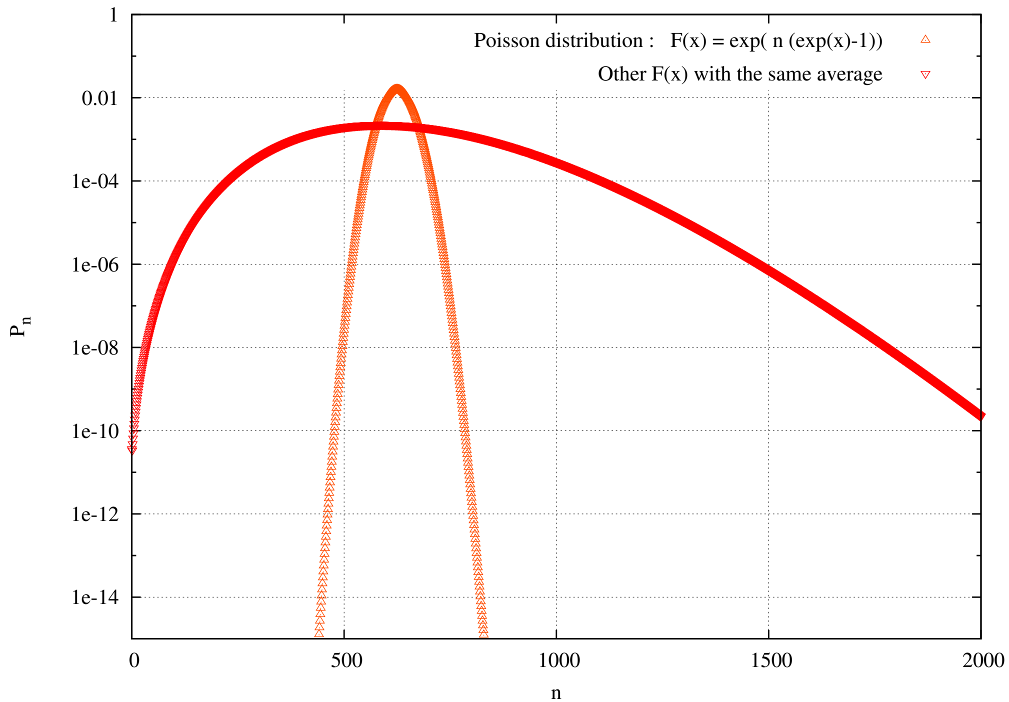
- At tree level, all the disconnected graphs are of order $1/g^2$
  - ▷ therefore, no truncation is possible
- The uncut vacuum-vacuum diagrams on both sides exponentiate into : $\exp(i\sum_{\rm conn} V)\;\exp(-i\sum_{\rm conn} V^*) = \exp(-\sum_r b_r/g^2)$

Multiplicity distribution

- Assume for a moment that we know the generating function $F(x)$. We can get the probability distribution as follows :

$$P_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; e^{-in\theta} \; F(i\theta)$$

Note : this is trivial to evaluate numerically by a FFT :



Multiplicity distribution

- So far, no practical method is known for calculating the generating function $F(x)$ given the current j, not even at tree level...
- How are we ever going to calculate some physical quantities in this theory?
- Even if $\exp(iV[j])\ \exp(-iV^*[j])$ is not calculable, we have :

$$e^{\mathcal{D}[j_+,j_-]}\ e^{iV[j_+]}\ e^{-iV^*[j_-]}\Big|_{j_+=j_-=j} = 1$$

- Any quantity for which we can exploit this cancellation is going to be much easier to calculate :
  - This is not the case of the individual P_n's
  - But this simplification happens for the moments

Calculation of the moments

- The formula $\langle n^p \rangle_{\rm conn} = \sum_r r^p b_r / g^2$ is very impractical
- Instead, go back to :

$$F(x) = e^{e^x \mathcal{D}[j_+, j_-]} \; e^{iV[j_+]} \; e^{-iV^*[j_-]} \Big|_{j_+=j_-=j}$$

- The average multiplicity is given by :

$$\langle n \rangle = F'(0) = \mathcal{D}[j_+, j_-] \; e^{iV_c[j_+, j_-]} \Big|_{j_+=j_-=j}$$

where $iV_c[j_+, j_-]$ is the sum of all the cut connected vacuum-vacuum diagrams, with j_+ on the $+$ side of the cut and j_- on the $-$ side of the cut

- More explicitly, this reads :

$$\langle n \rangle = \frac{1}{Z} \int_{x,y} G^0_{+-}(x,y) \; (\Box_x + m^2)(\Box_y + m^2) \; \left[\frac{\delta iV_c}{\delta j_+(x)} \frac{\delta iV_c}{\delta j_-(y)} + \frac{\delta^2 iV_c}{\delta j_+(x) \delta j_-(y)} \right]$$

Calculation of the moments

- The functional derivatives of $iV_c[j_+, j_-]$ with respect to $j_\pm$ give Green's functions that are calculated like cut diagrams, but with external legs of type $+$ or $-$. Let us denote :

$$\Gamma_\pm(x) \equiv \frac{\square_x + m^2}{Z} \left. \frac{\delta i V_c}{\delta j_\pm(x)} \right|_{j_+=j_-=j}$$

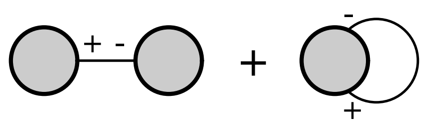
$$\Gamma_{+-}(x,y) \equiv \frac{(\square_x + m^2)(\square_y + m^2)}{Z^2} \left. \frac{\delta^2 i V_c}{\delta j_+(x) \delta j_-(y)} \right|_{j_+=j_-=j}$$

- Therefore, we have :

$$\langle n \rangle = \int_{x,y} Z G^0_{+-}(x,y) \left[\Gamma_+(x)\Gamma_-(y) + \Gamma_{+-}(x,y) \right]$$

- Or, diagrammatically :

$\langle n \rangle =$



- $\langle n \rangle$ is a sum of simply connected graphs

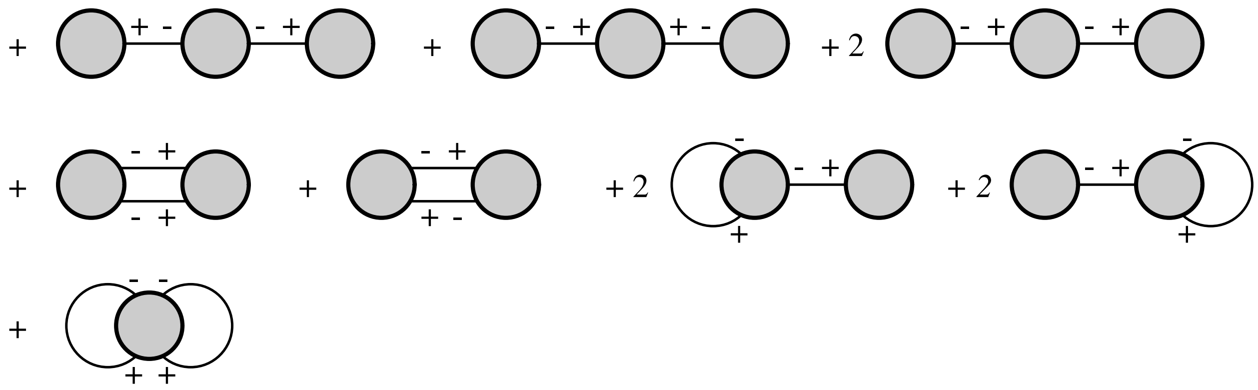
Calculation of the moments

- The same method can be applied to the calculation of the variance :

$$\left<n^2\right> - \left<n\right>^2 = \frac{d^2}{dx^2}\ln(F(x))_{x=0}$$
$$= \left[\mathcal{D}[j_+,j_-] + \mathcal{D}^2[j_+,j_-]\right] \; e^{iV_c[j_+,j_-]}\Big|_{\substack{j_+=j_-=j \\ \text{connected}}}$$

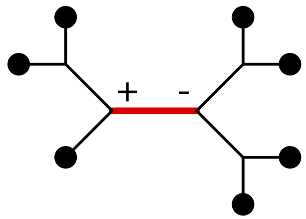
- In terms of diagrams :

$$\left<n^2\right> - \left<n\right>^2 = \left<n\right>$$



Leading Order

- At leading order – i.e. $\mathcal{O}(1/g^2)$ – the shaded blobs in the formula for $\langle n \rangle$ must be evaluated at tree level (and the second diagram does not contribute) :

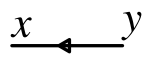


- For all the vertices except the two which are labelled explicitly, we must sum over the indices $+/-$
- We must also sum over all the topologies for the tree diagrams on the left and on the right of the G^0_{+-} propagator

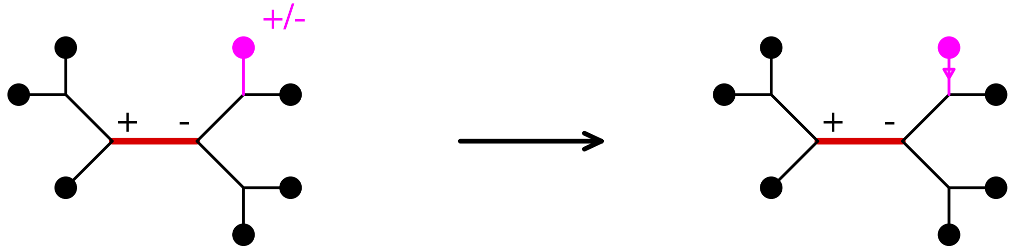
Leading Order

- The sum over the $\pm$ indices attached to the vertices in each of the tree diagrams can be performed by noting that :

$$\text{For } \epsilon = +, - \quad , \qquad G^0_{\epsilon+}(x,y) - G^0_{\epsilon-}(x,y) = G^0_R(x,y)$$

where $G^0_R(x,y)$ is the free retarded propagator, denoted 

- Starting from the “leaves” of the trees, use this property recursively to replace all the $G^0_{\pm\pm}$ propagators by retarded propagators :



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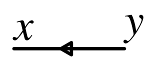
where $G^0_R(x,y)$ is the free **retarded propagator**, denoted 

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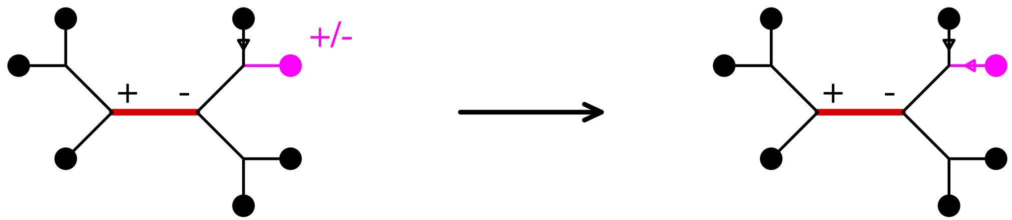

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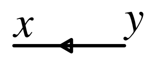
where $G^0_R(x,y)$ is the free retarded propagator, denoted 

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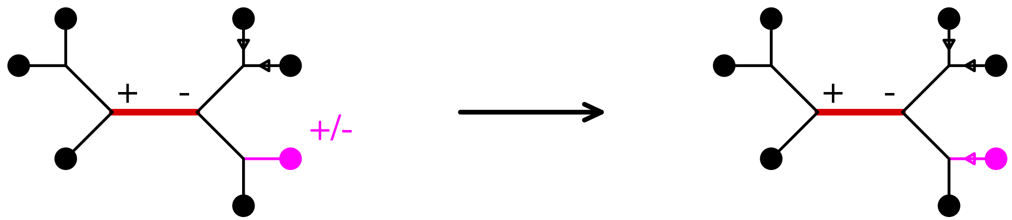

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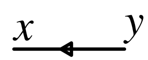
where $G^0_R(x,y)$ is the free **retarded propagator**, denoted 

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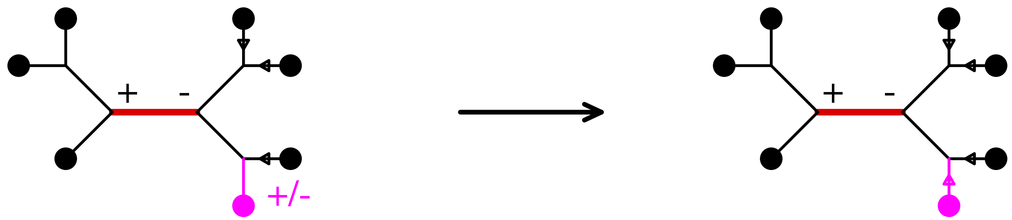

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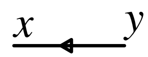
where $G^0_R(x,y)$ is the free retarded propagator, denoted 

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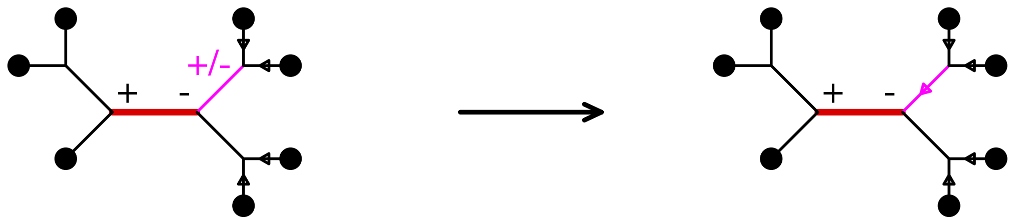

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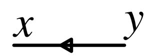
where $G^0_R(x,y)$ is the free retarded propagator, denoted 

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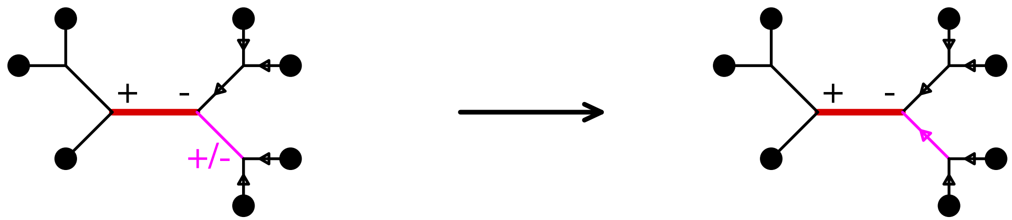

- After having done the same transformation on the other half of the diagram, all the propagators except the intermediate one have been replaced by retarded propagators

Classical solution

- The classical equation of motion reads :

$$\left(\square + m^2\right) \phi(x) + \frac{g}{2}\phi^2(x) = j(x)$$

- The retarded solution of this equation, with the boundary condition $\phi(x) = 0$ when $x_0 \to -\infty$, can be found iteratively in g : $\phi = \phi_{(0)} + \phi_{(1)} + \cdots$, by rewriting the EOM as :

$$\phi(x) = \int d^4y\, G_R^0(x-y) \left[-i\,\frac{g}{2}\phi^2(y) + i\,j(y)\right]$$

where $G_R^0(x-y)$ is the retarded Green's function for the operator $\square + m^2$, normalized by :

$$\left(\square_x + m^2\right) G_R^0(x-y) = -i\,\delta(x-y)$$

or, in momentum space, $G_R^0(p) = \dfrac{i}{p^2 - m^2 + ip_0\epsilon}$

Classical solution

- Order g^0 :

$$\phi_{(0)}(x) = \int d^4y\, G_R^0(x-y)\, i\, j(y)$$

- Order g^1 :

$$\phi_{(0)}(x) + \phi_{(1)}(x) = \int d^4y\, G_R^0(x-y)\left[-i\,\frac{g}{2}\phi_{(0)}^2(y) + i\, j(y)\right]$$

i.e.

$$\phi_{(1)}(x) = -i\,\frac{g}{2}\int d^4y\, G_R^0(x-y)\left[\int d^4z\, G_R^0(y-z)\, i\, j(z)\right]^2$$

- One can construct the solution iteratively, by using in the r.h.s. the solution found in the previous orders

Classical solution

- Therefore, the classical solution can be represented as :

Classical solution

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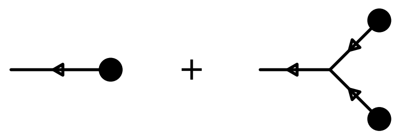

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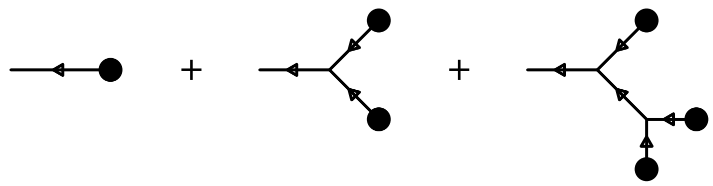

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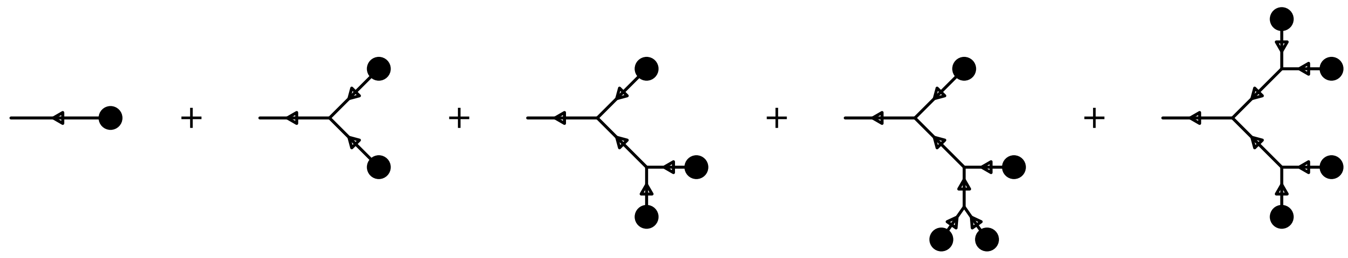

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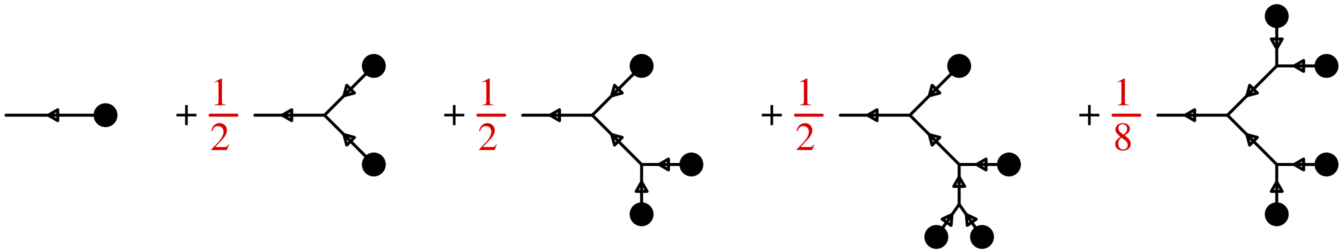

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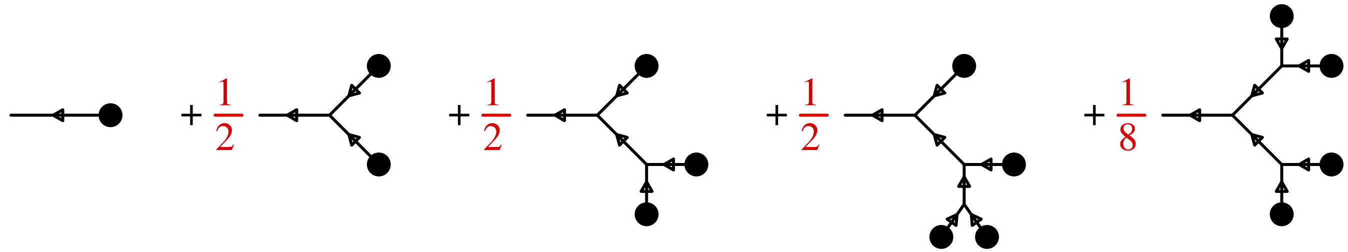


- The classical solution is given by the sum of all the tree diagrams with retarded propagators. It resums all the powers of g that are accompanied by a source j

- The quantity that appears in $\langle n \rangle_{_{LO}}$ does not have the last retarded propagator. Therefore, it is :

$$(\square_x + m^2)\phi_c(x)$$

Leading Order (cont.)

- Finally, one gets $\left\langle n\right\rangle_{{}_{LO}}$ in terms of the retarded solution ϕ_c of the EOM :

$$E_p \left.\frac{d\langle n\rangle}{d^3\vec{\boldsymbol{p}}}\right|_{{}_{LO}} = \frac{1}{16\pi^3}\left|(p^2-m^2)\phi_c(p)\right|^2$$

- $(p^2-m^2)\phi_c(p)$ is given by a 4-dim Fourier transform :

$$(p^2-m^2)\phi_c(p) = -\int d^4x\; e^{ip\cdot x}\;(\square_x+m^2)\phi_c(x)$$

- This formula is cumbersome in practice because it requires to store the solution of the EOM at all times. Instead, write :

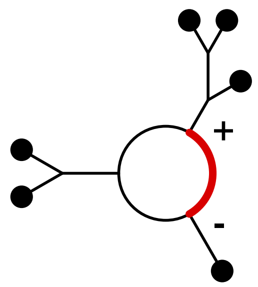
$$e^{ip\cdot x}\left[\partial^2_{x_0}+E_p^2\right]\phi_c(x) = \partial_{x_0}\; e^{ip\cdot x}\;\left[\partial_{x_0}-iE_p\right]\phi_c(x)$$

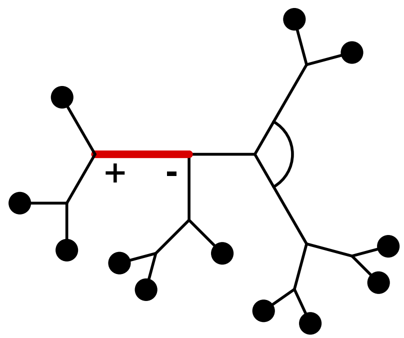
from which one can obtain :

$$(p^2-m^2)\phi_c(p) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\;\left[\partial_{x_0}-iE_p\right]\phi_c(x)$$

Next to Leading Order

- There are two types of corrections at NLO :

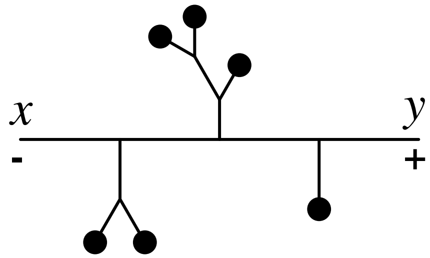




- They both contribute at order g^0. The first type of NLO topologies would in fact be the leading contribution for quark production
  ▷ we consider only this one in the following (but the other one can be calculated as well)

Next to Leading Order

- We need to calculate the sum of the following tree diagrams :



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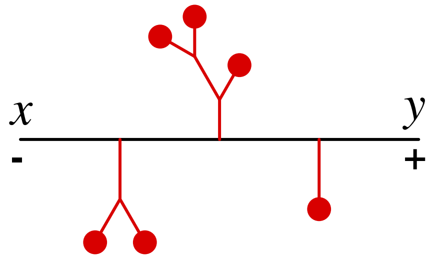
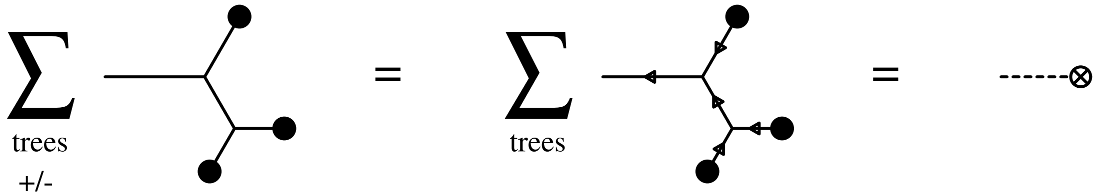


- One can perform a partial resummation of all the sub-diagrams that correspond to the classical solution :



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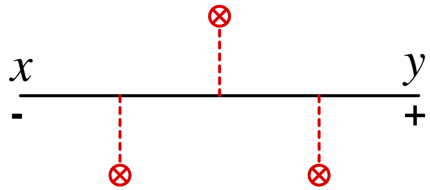

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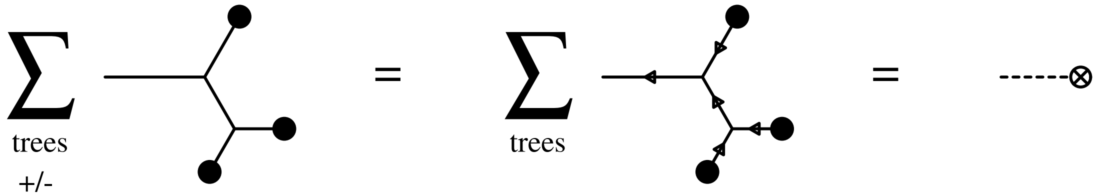


- Thus, we need the full tree level propagator $G_{-+}(x, y)$ in the presence of the retarded background field ϕ_c. Note : the classical field insertion is the same for the $+$ and $-$ indices

Next to Leading Order

- The summation of all the classical field insertions can be done via a Lippmann-Schwinger equation :

$$G_{\epsilon\epsilon'}(x,y) = G^0_{\epsilon\epsilon'}(x,y) - ig \sum_{\eta,\eta'=\pm} \int d^4z\, G^0_{\epsilon\eta}(x,z)\phi_c(z)\boldsymbol{\sigma}^3_{\eta\eta'} G_{\eta'\epsilon'}(z,y)$$

- This equation is rather non-trivial to solve in this form, because of the mixing of the 4 components of the propagator. Perform a rotation on the $\pm$ indices :

$$G_{\epsilon\epsilon'} \quad \rightarrow \quad \boldsymbol{G}_{\alpha\beta} \equiv \sum_{\epsilon,\epsilon'=\pm} U_{\alpha\epsilon}U_{\beta\epsilon'}G_{\epsilon\epsilon'}$$

$$\boldsymbol{\sigma}^3_{\epsilon\epsilon'} \quad \rightarrow \quad \boldsymbol{\Sigma}^3_{\alpha\beta} \equiv \sum_{\epsilon,\epsilon'=\pm} U_{\alpha\epsilon}U_{\beta\epsilon'}\boldsymbol{\sigma}^3_{\epsilon\epsilon'}$$

- A useful choice for the rotation matrix U is $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Next to Leading Order

- Under this rotation, the matrix propagator and field insertion become :

$$\boldsymbol{G}_{\alpha\beta} = \begin{pmatrix} 0 & G_{A} \\ G_{R} & G_{S} \end{pmatrix} \quad , \qquad \boldsymbol{\Sigma}^3_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $G^0_{S}(p) = 2\pi\delta(p^2 - m^2)$

- The main simplification comes from the fact that $\boldsymbol{G}^0\boldsymbol{\Sigma}^3$ is the sum of a diagonal matrix and a nilpotent matrix
- One finds that G_{R} and G_{A} do not mix, i.e. they obey the equations :

$$G_{R,A}(x,y) = G^0_{R,A}(x,y) - i\,g\int d^4z\, G^0_{R,A}(x,z)\phi_c(z)G_{R,A}(z,y)$$

- One can express G_{S} in terms of G_{R} and G_{A} :

$$G_{S} = G_{R} * {G^0_{R}}^{-1} * G^0_{S} * {G^0_{A}}^{-1} * G_{A}$$

Next to Leading Order

- In order to go back to G_{-+}, invert the rotation :

$$G_{-+} = \frac{1}{2}\left[G_A - G_R + G_S\right]$$

- Split $G_{R,A}$ into free propagators and a scattering matrix :

$$G_{R,A} \equiv G^0_{R,A} + G^0_{R,A} * T_{R,A} * G^0_{R,A}$$

Note : the retarded/advanced scattering matrices $T_{R,A}$ obey :

$$T_R - T_A = T_R * \left[G^0_R - G^0_A\right] * T_A$$

- Wrapping up everything, in momentum space, gives :

$$E_p \left.\frac{d\langle n\rangle}{d^3\vec{\boldsymbol{p}}}\right|_{NLO} = \frac{1}{16\pi^3}\int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q}\left|T_R(p,-q)\right|^2$$

▷ One has to obtain the retarded propagator in the classical field ϕ_c, amputate the external legs, square and integrate over the (on-shell) momentum at one end

Next to Leading Order

- $T_R(p,-q)$ can be obtained from retarded solutions of the EOM of a small fluctuation on top of ϕ_c

$$\big(\square + m^2 + g\,\phi_c(x)\big)\eta(x) = 0$$

- Start from Green's formula for the retarded solution $\eta(x)$:

$$\eta(x) = \int d^3\vec{\boldsymbol{y}}\; G_R(x,y)\;\overset{\leftrightarrow}{\partial}_{y_0}\;\eta(y)$$

- From there, it is straightforward to verify that :

$$T_R(p,-q) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\left[\partial_{x_0} - iE_p\right]\eta(x)$$

with $\eta(x) = e^{iq\cdot x}$ when $x^0 \to -\infty$

- In other words, one must solve the equation of propagation of small fluctuations on top of the classical field, with a plane wave as the initial condition

Gluon production

Krasnitz, Venugopalan (1998), Lappi (2003)

- At tree level, the gluon spectrum is given directly by the retarded solution of Yang-Mills equations :

$$E_p \frac{d\langle n_g \rangle}{d^3\vec{p}} = \frac{1}{16\pi^3} \sum_\lambda \left| \lim_{x^0 \to +\infty} \int d^3\vec{x}\; e^{ip\cdot x}\; [\partial_{x_0} - iE_p]\, \epsilon_\mu^{(\lambda)}(\vec{p})\, A^\mu(x) \right|^2$$

- The calculation is usually done in the gauge :

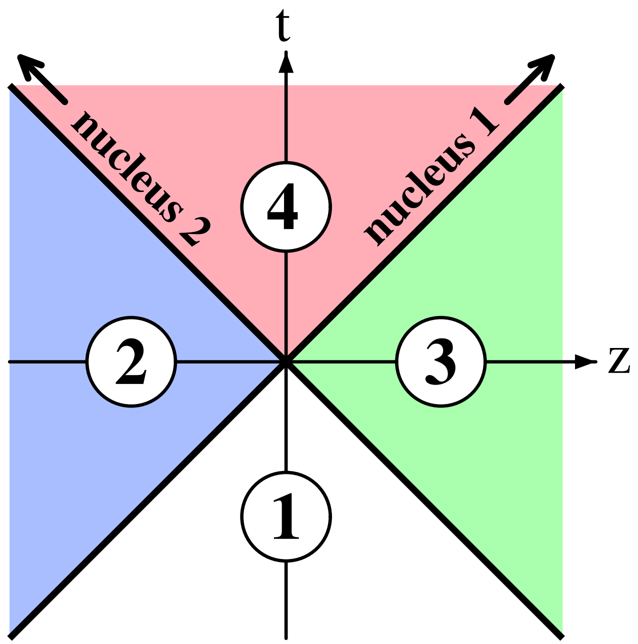
$$A^\tau = x^+ A^- + x^- A^+ = 0$$

  - This gauge interpolates between two light-cone gauges : $A^- = 0$ on the trajectory $z = t$ and $A^+ = 0$ on the trajectory $z = -t$
  - This implies that the produced gauge field does not make the currents J^+, J^- precess in color space

- In this gauge, it is easy to find the field at $\tau = 0^+$, and then let it evolve according to the vacuum Yang-Mills equations (because the currents are zero at $\tau > 0$)

Classical color field

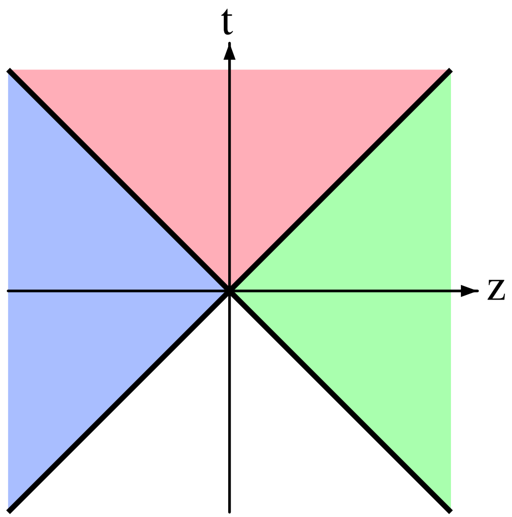
- Space-time structure of the classical color field:



  - Region 1 : no causal relation to either nuclei
  - Region 2 : causal relation to the 1st nucleus only
  - Region 3 : causal relation to the 2nd nucleus only
  - Region 4 : causal relation to both nuclei

Classical color field

- Propagation through region 1:

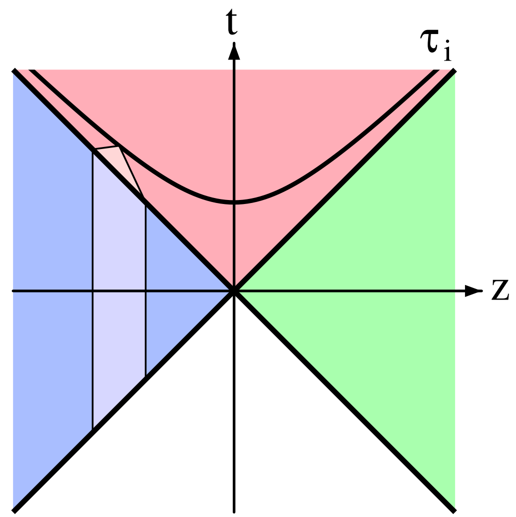


▷ trivial : the classical field is entirely determined by the initial condition, i.e.

$$A^\mu = 0$$

Classical color field

- Propagation through region 2:



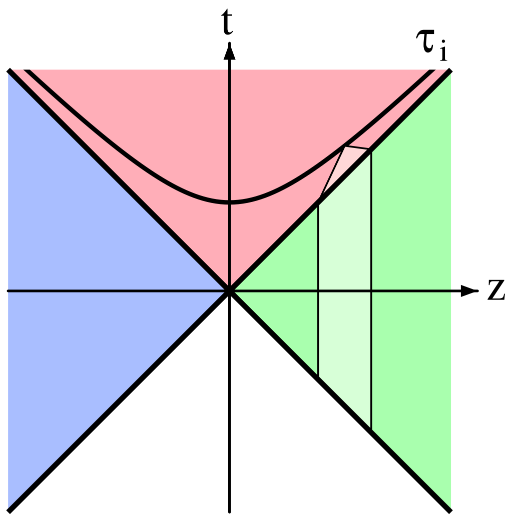
▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

$$A^+ = A^- = 0 \quad , \quad A^i = \theta(x^-)\,\frac{i}{g}U_1(\vec{\boldsymbol{x}}_\perp)\partial^i U_1^\dagger(\vec{\boldsymbol{x}}_\perp)$$

with $\quad U_1(\vec{\boldsymbol{x}}_\perp) = T_+ \exp\; ig\int dx^+ T^a \frac{1}{\boldsymbol{\nabla}_\perp^2}\rho_1^a(x^+,\vec{\boldsymbol{x}}_\perp)$

Classical color field

- Propagation through region 3:



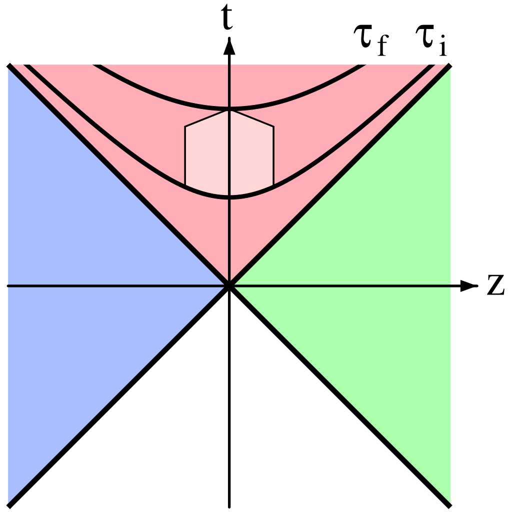
▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

$$A^+ = A^- = 0 \quad , \quad A^i = \theta(x^+)\,\frac{i}{g}U_2(\vec{\boldsymbol{x}}_\perp)\partial^i U_2^\dagger(\vec{\boldsymbol{x}}_\perp)$$

$$\text{with} \quad U_2(\vec{\boldsymbol{x}}_\perp) = T_-\ \exp\ ig\int dx^- T^a \frac{1}{\boldsymbol{\nabla}_\perp^2}\rho_2^a(x^-,\vec{\boldsymbol{x}}_\perp)$$

Classical color field

- Propagation through region 4:



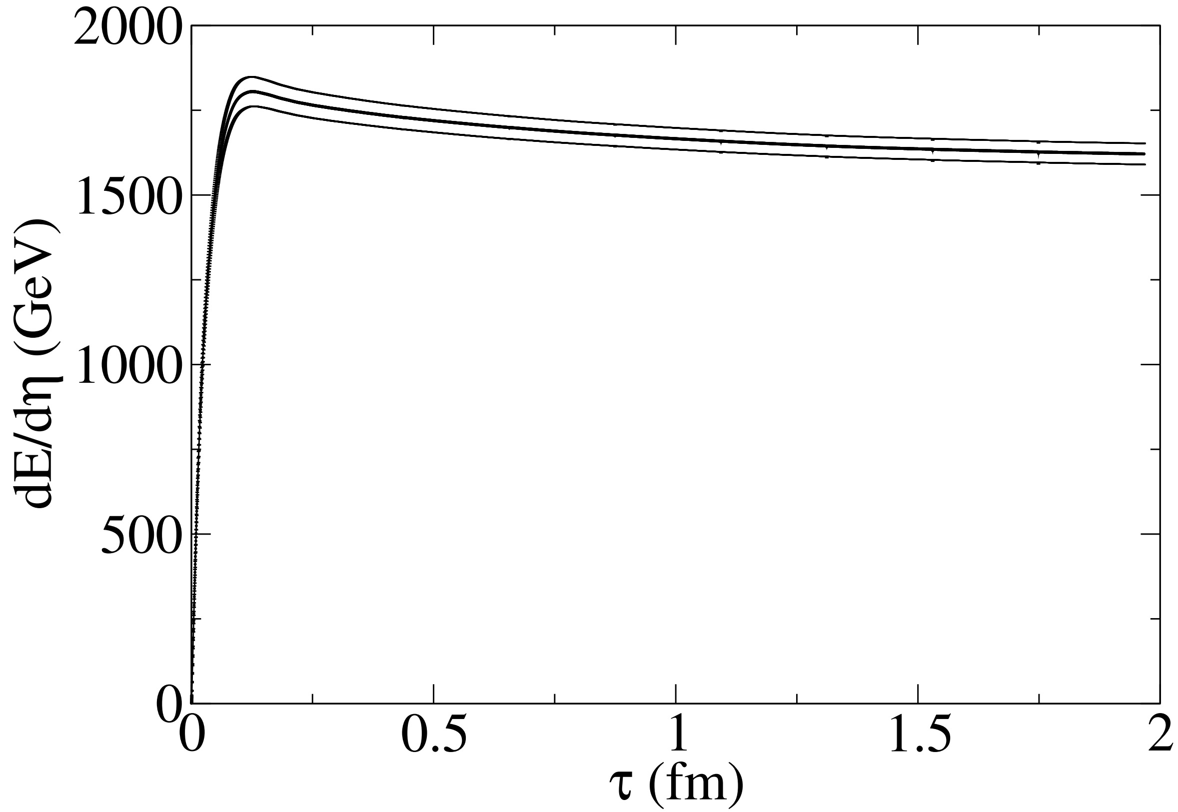
▷ one must solve numerically the Yang-Mills equations with the following initial condition at $\tau_i = 0^+$:

$$A^i(\tau=0,\vec{\boldsymbol{x}}_\perp) = \frac{i}{g}\Big(U_1(\vec{\boldsymbol{x}}_\perp)\partial^i U_1^\dagger(\vec{\boldsymbol{x}}_\perp) + U_2(\vec{\boldsymbol{x}}_\perp)\partial^i U_2^\dagger(\vec{\boldsymbol{x}}_\perp)\Big)$$

$$A^\eta(\tau=0,\vec{\boldsymbol{x}}_\perp) = -\frac{i}{2g}\Big[U_1(\vec{\boldsymbol{x}}_\perp)\partial^i U_1^\dagger(\vec{\boldsymbol{x}}_\perp)\ ,\ U_2(\vec{\boldsymbol{x}}_\perp)\partial^i U_2^\dagger(\vec{\boldsymbol{x}}_\perp)\Big]$$

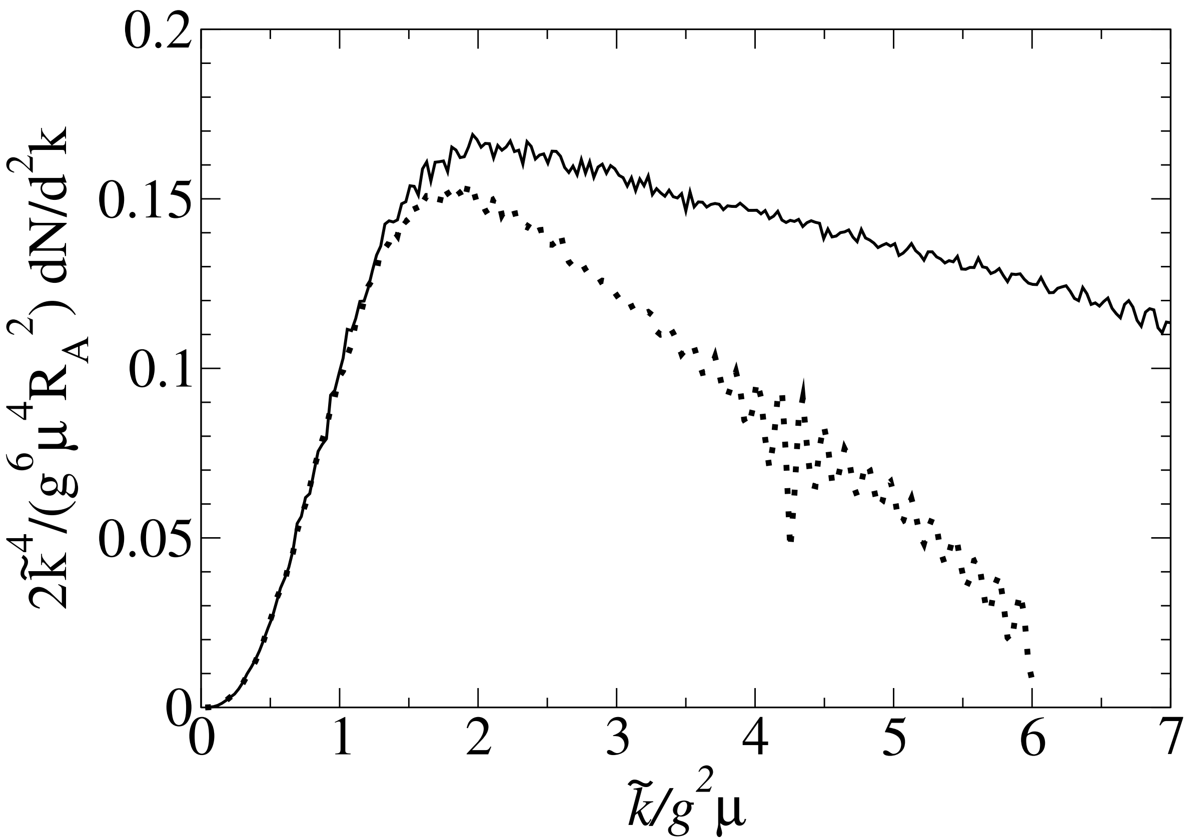
Energy per unit rapidity

- Time dependence of $dE/d\eta$:



Gluon spectrum

- Gluon spectra for 512^2 and 256^2 lattices:



- Lattice artifacts at large momentum (does not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD

Quark production

FG, Kajantie, Lappi (2004, 2005)

- The inclusive quark spectrum can be obtained from the retarded propagator of the quark in the classical color field:

$$E_p \frac{d\langle n_{\rm q}\rangle}{d^3\vec{\boldsymbol{p}}} = \frac{1}{16\pi^3}\int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q}\,\left|\overline{u}(\vec{\boldsymbol{p}})T_R(p,-q)v(\vec{\boldsymbol{q}})\right|^2$$

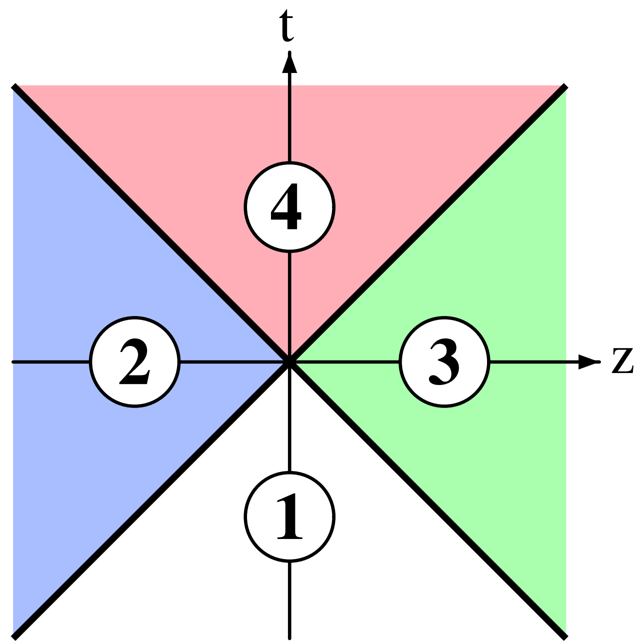
- Alternate representation of the retarded amplitude:

$$\overline{u}(\vec{\boldsymbol{p}})T_R(p,-q)v(\vec{\boldsymbol{q}}) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\,u^\dagger(\vec{\boldsymbol{p}})\psi_{\boldsymbol{q}}(x)$$

$$(i\partial\!\!\!/_x - gA\!\!\!/(x) - m)\psi_{\boldsymbol{q}}(x) = 0\,,\;\; \psi_{\boldsymbol{q}}(x^0,\vec{\boldsymbol{x}}) \underset{x^0\to-\infty}{\longrightarrow} v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

Background field

- Space-time structure of the classical color field:

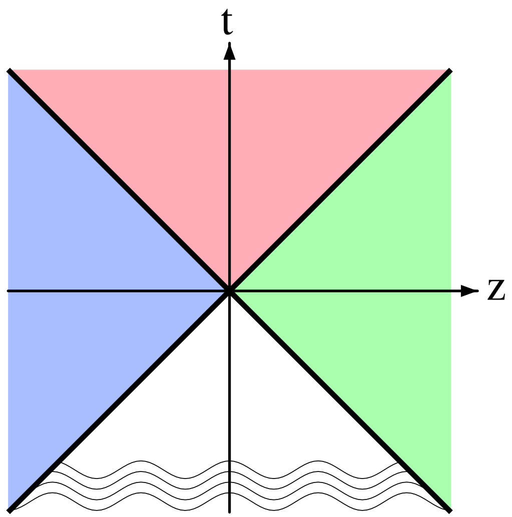


- Region 1: $A^\mu = 0$
- Region 2: $A^\pm = 0$, $A^i = \frac{i}{g} U_1 \nabla_\perp^i U_1^\dagger$
- Region 3: $A^\pm = 0$, $A^i = \frac{i}{g} U_2 \nabla_\perp^i U_2^\dagger$
- Region 4: $A^\mu \neq 0$

- **Notes:**
  - In the region 4, A^μ is known only numerically
  - We will have to solve the Dirac equation numerically as well

Quark propagation

- Propagation through region 1:

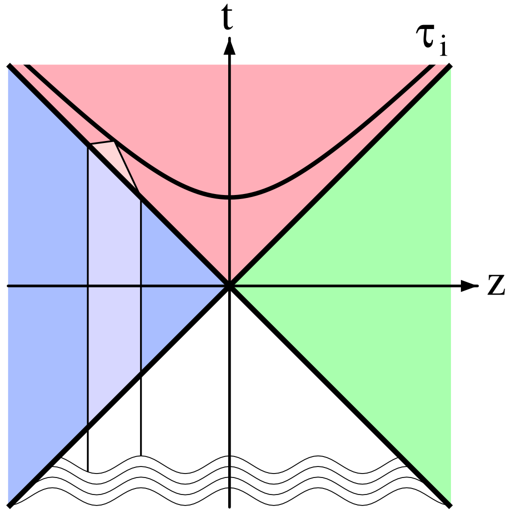


▷ trivial because there is no background field

$$\psi_{\boldsymbol{q}}(x) = v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

Quark propagation

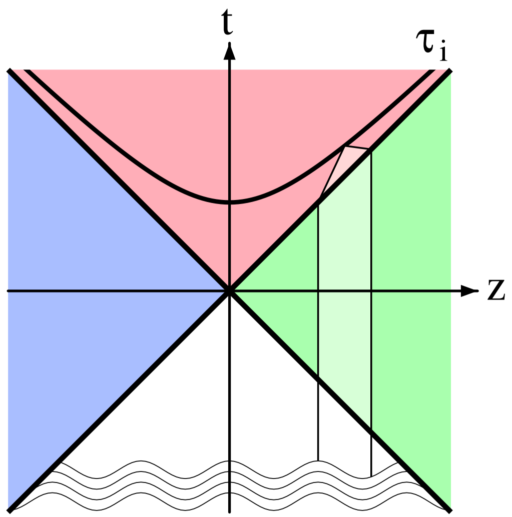
- Propagation through region 2:



- ▷ Pure gauge background field
  - ▷ $\psi_{\boldsymbol{q}}^{-}(\tau_i)$ can be obtained analytically

Quark propagation

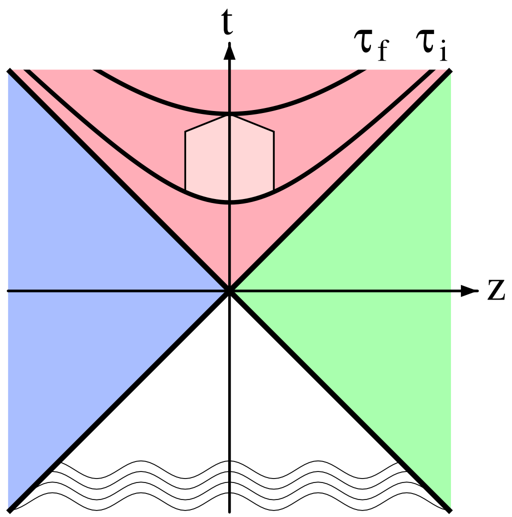
- Propagation through region 3:



  - ▷ Pure gauge background field
    - ▷ $\psi^+_{\boldsymbol{q}}(\tau_i)$ can be obtained analytically

Quark propagation

- Propagation through region 4:



▷ One must solve the Dirac equation :

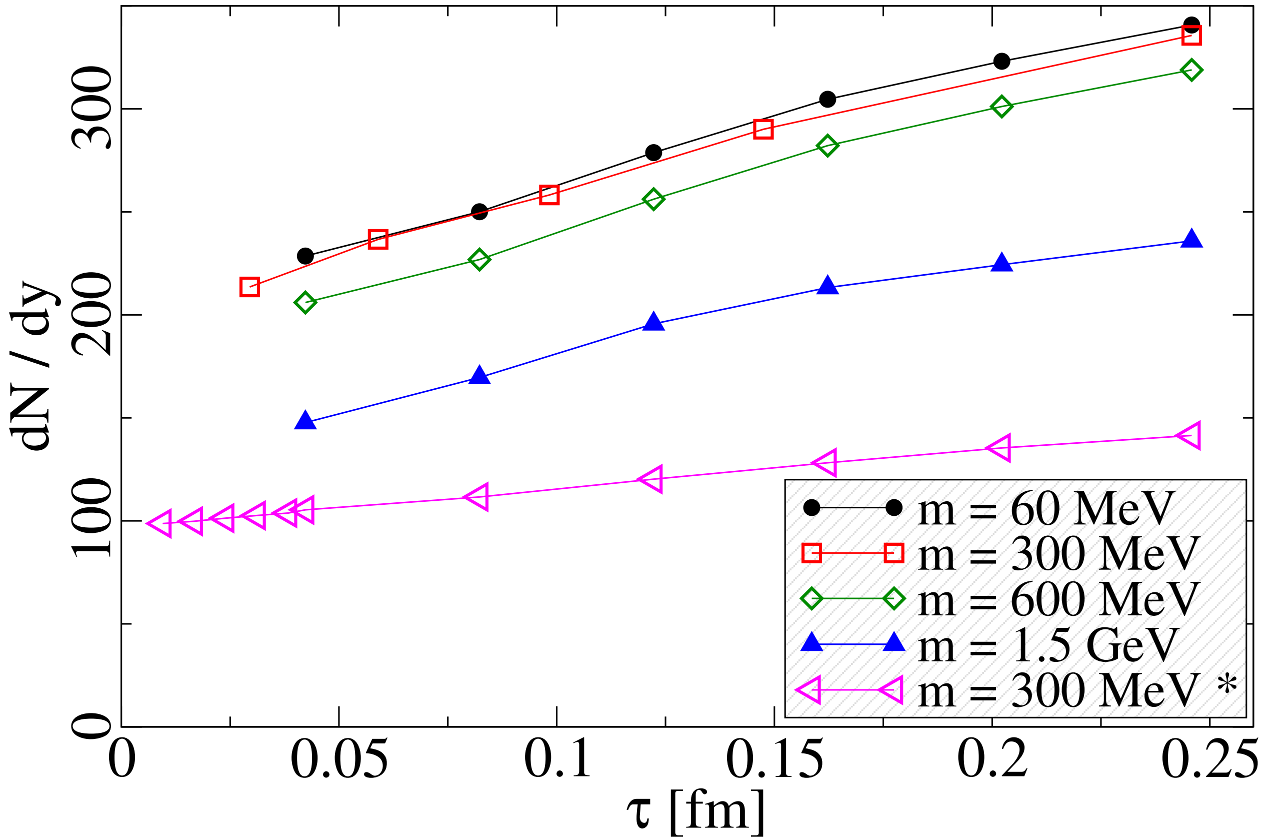
$$\left[i\partial\!\!\!/ - gA\!\!\!/ - m\right] \psi_{\boldsymbol{q}}(\tau, \eta, \vec{\boldsymbol{x}}_\perp) = 0$$

▷ initial condition: $\psi_{\boldsymbol{q}}(\tau_i) = \psi^+_{\boldsymbol{q}}(\tau_i) + \psi^-_{\boldsymbol{q}}(\tau_i)$

($\tau_i = 0^+$ in practice)

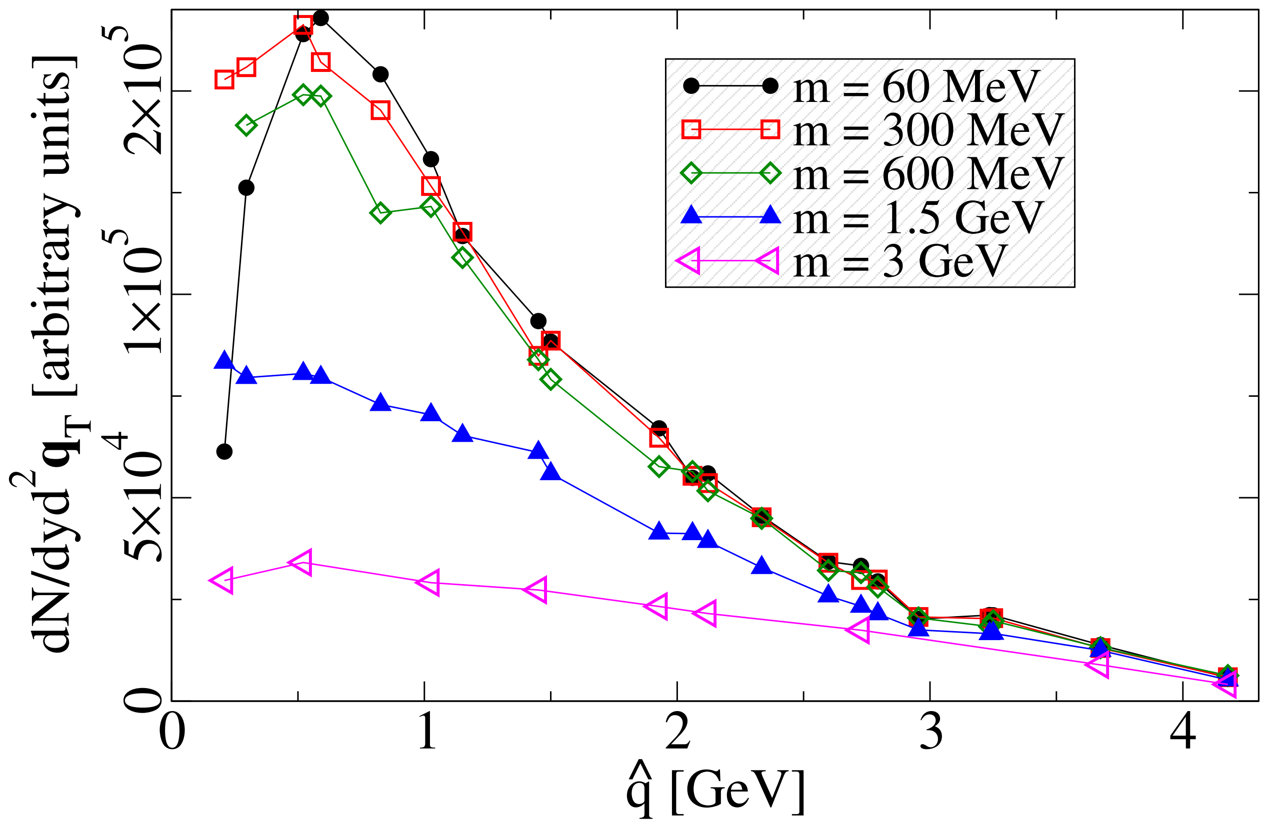
Time dependence

- $g^2\mu = 2$ GeV , (*) $g^2\mu = 1$ GeV :



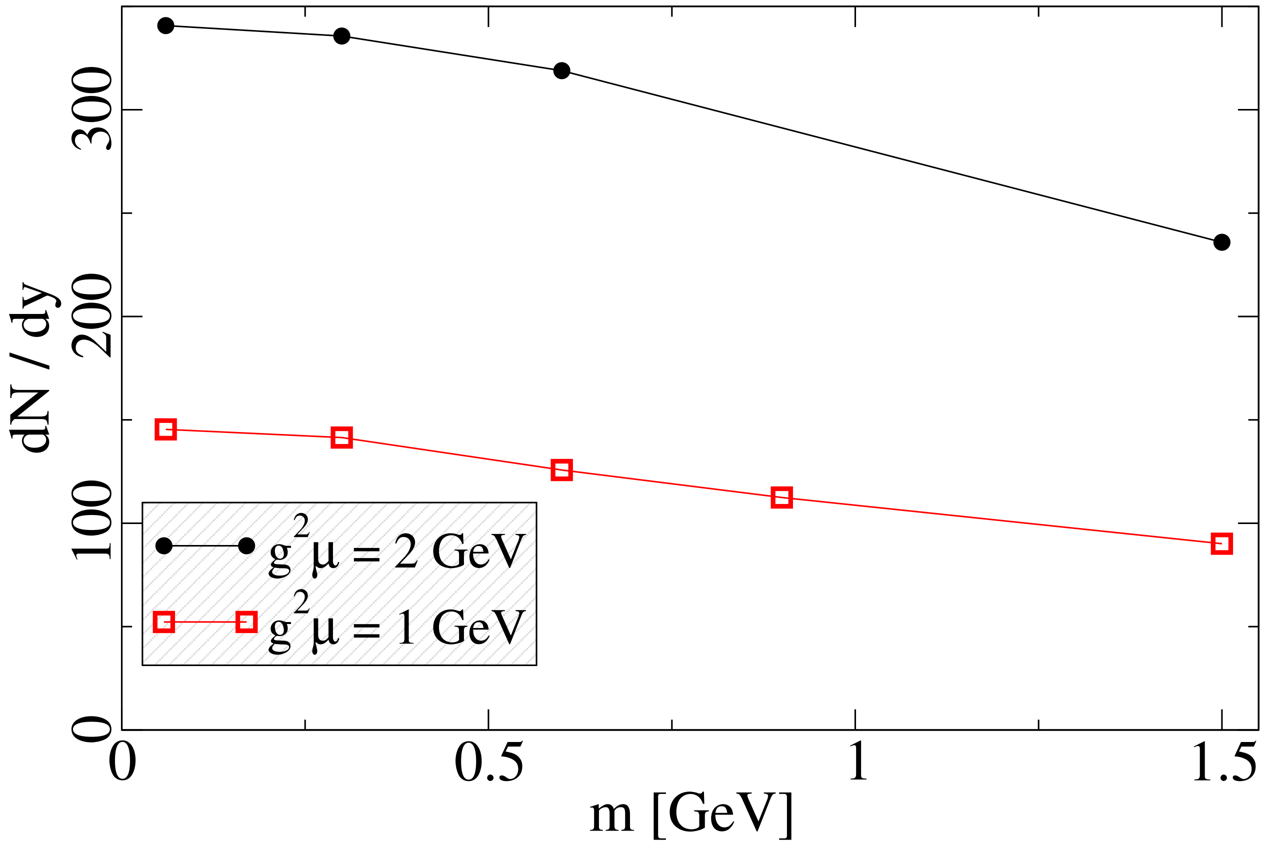
Spectra for various quark masses

- $g^2\mu = 2$ GeV , $\tau = 0.25$ fm :



Mass dependence of dN/dy

- Number of quarks at $\tau = 0.25$ fm :



g2mu dependence of dN/dy

- Number of quarks at $\tau = 0.25$ fm :