High energy hadronic interactions in QCD and applications to heavy ion collisions

II – Lessons from Deep Inelastic Scattering

François Gelis

CEA / DSM / SPhT

General outline

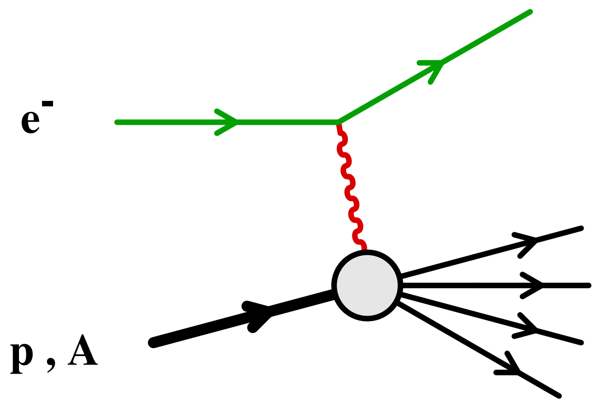
- Lecture I : Introduction and phenomenology
- Lecture II : Lessons from Deep Inelastic Scattering
- Lecture III : QCD on the light-cone
- Lecture IV : Saturation and the Color Glass Condensate
- Lecture V : Calculating observables in the CGC

Lecture II : Lessons from DIS

- Kinematics of Deep Inelastic Scattering
- Structure functions
- Experimental facts
- Naive parton model
- Light-cone behavior of a free field theory
- Scaling violations
- Factorization

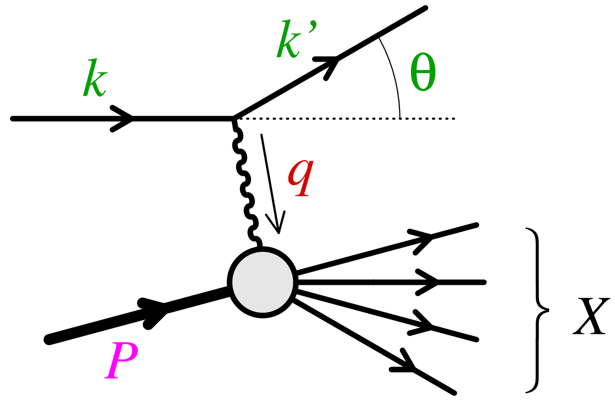
Introduction to DIS

- Basic idea : smash a well known probe on a nucleon or nucleus in order to try to figure out what is inside...
- Photons are very well suited for that purpose because their interactions are well understood
- Deep Inelastic Scattering : collision between an electron and a nucleon or nucleus, by exchange of a virtual photon



- Variant : collision with a neutrino, by exchange of $Z^0, W^\pm$

Kinematical variables



- Note : the virtual photon is spacelike: $q^2 \leq 0$
- Other invariants of the reaction :

$$\begin{aligned} \nu &\equiv P \cdot q \\ s &\equiv (P+k)^2 \\ M_X^2 &\equiv (P+q)^2 = m_N^2 + 2\nu + q^2 \end{aligned}$$

- One uses commonly : $Q^2 \equiv -q^2$ and $x \equiv Q^2/2\nu$
- In general $M_X^2 \geq m_N^2$, and we have : $0 \leq x \leq 1$ ($x = 1$ corresponds to the case of elastic scattering)

DIS cross-section

- The simplest cross-section is the inclusive cross-section, obtained by measuring the momentum of the scattered electron and summing over all the hadronic final states X

$$E' \frac{d\sigma_{e^-N}}{d^3\vec{\boldsymbol{k}}'} = \sum_{\text{states } X} E' \frac{d\sigma_{e^-N\to e^-X}}{d^3\vec{\boldsymbol{k}}'}$$

$$E' \frac{d\sigma_{e^-N\to e^-X}}{d^3\vec{\boldsymbol{k}}'} = \int \frac{[d\Phi_X]}{32\pi^3(s-m_N^2)} (2\pi)^4 \delta(P+k-k'-P_X) \left\langle |\mathcal{M}_X|^2 \right\rangle_{\text{spin}}$$

$$\mathcal{M}_X = \frac{ie}{q^2} \left[\overline{u}(\vec{\boldsymbol{k}}')\gamma^\mu u(\vec{\boldsymbol{k}})\right] \left\langle X \middle| J_\mu(0) \middle| N(P) \right\rangle$$

- In the amplitude squared appears the leptonic tensor :

$$L^{\mu\nu} \equiv \left\langle \overline{u}(\vec{\boldsymbol{k}}')\gamma^\mu u(\vec{\boldsymbol{k}})\overline{u}(\vec{\boldsymbol{k}})\gamma^\nu u(\vec{\boldsymbol{k}}') \right\rangle_{\text{spin}}$$
$$= 2(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}\, k \cdot k')$$

(the electron mass has been neglected)

DIS cross-section

- The inclusive cross-section can be written as :

$$E' \frac{d\sigma_{e-N}}{d^3\vec{\boldsymbol{k}}'} = \frac{1}{32\pi^3(s - m_N^2)} \frac{e^2}{q^4} 4\pi L^{\mu\nu} W_{\mu\nu}$$

where $W_{\mu\nu}$ is the hadronic tensor, defined as:

$$4\pi W_{\mu\nu} \equiv \sum_{\text{states } X} \int [d\Phi_X] (\boldsymbol{2\pi})^4 \delta(\boldsymbol{P} + \boldsymbol{q} - \boldsymbol{P_X}) \times \Big\langle \big\langle N(P) \big| J_\nu^\dagger(0) \big| X \big\rangle \big\langle X \big| J_\mu(0) \big| N(P) \big\rangle \Big\rangle_{\text{spin}}$$

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- $W_{\mu\nu}$ contains all the informations about the properties of the nucleon under consideration that are relevant to the interaction with the photon
- This object cannot be calculated perturbatively
- It obeys: $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$ (conservation of e.m. current)

Structure functions

- For a (spin-averaged) nucleon, the most general form of $W_{\mu\nu}$ is:

$$W_{\mu\nu} = -W_1 g_{\mu\nu} + W_2 \frac{P_\mu P_\nu}{m_N^2} + W_3 \epsilon_{\mu\nu\rho\sigma} \frac{P^\rho q^\sigma}{m_N^2} + W_4 \frac{q_\mu q_\nu}{m_N^2} + W_5 \frac{P_\mu q_\nu}{m_N^2} + W_6 \frac{q_\mu P_\nu}{m_N^2}$$

  - $W_3 = 0$ for parity conserving currents (like e.m. currents)
  - $W_{\mu\nu} = W_{\nu\mu}$ from parity and time-reversal symmetry hence $W_5 = W_6$
  - From the Ward identities $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$, one gets:

$$W_5 = -W_2 \frac{P \cdot q}{q^2}$$

$$W_4 = W_1 \frac{m_N^2}{q^2} + W_2 \frac{(P \cdot q)^2}{q^4}$$

Structure functions

- Therefore, for interactions with a photon, we have:

$$W_{\mu\nu} = -W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{m_N^2}\left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)$$

- And the DIS cross-section in the nucleon rest frame reads:

$$\frac{d\sigma_{e-N}}{dE'd\Omega} = \frac{\alpha_{\rm em}^2}{4m_N E^2\sin^4(\theta/2)}\left[2\sin^2(\theta/2)W_1 + \cos^2(\theta/2)W_2\right]$$

where Ω is the solid angle of the scattered electron

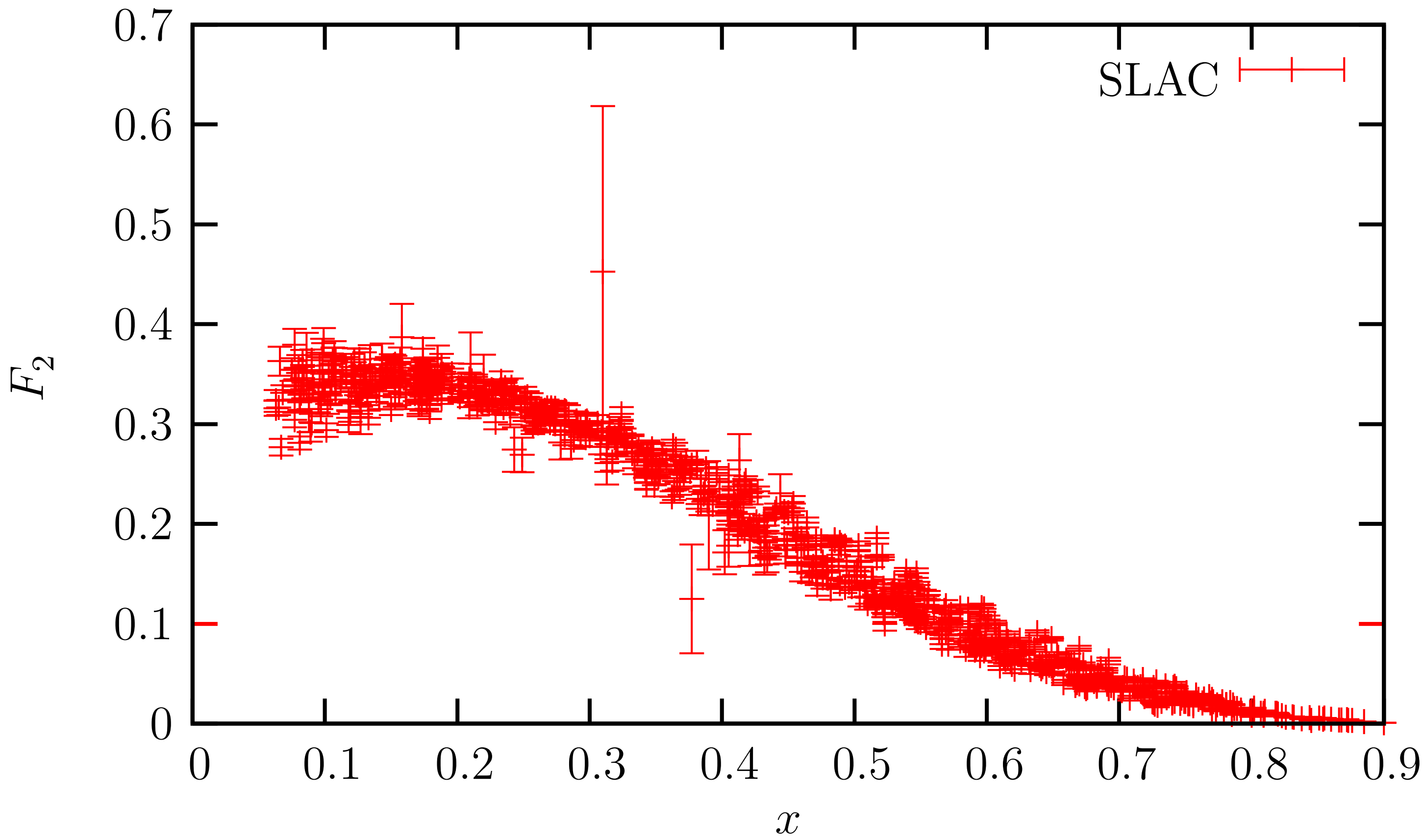
- It is customary to define slightly rescaled structure functions:

$$F_1 \equiv W_1 \quad , \quad F_2 \equiv \frac{\nu}{m_N^2}W_2$$

- Note: F_1 is proportional to the interaction cross-section between the nucleon and a transverse photon

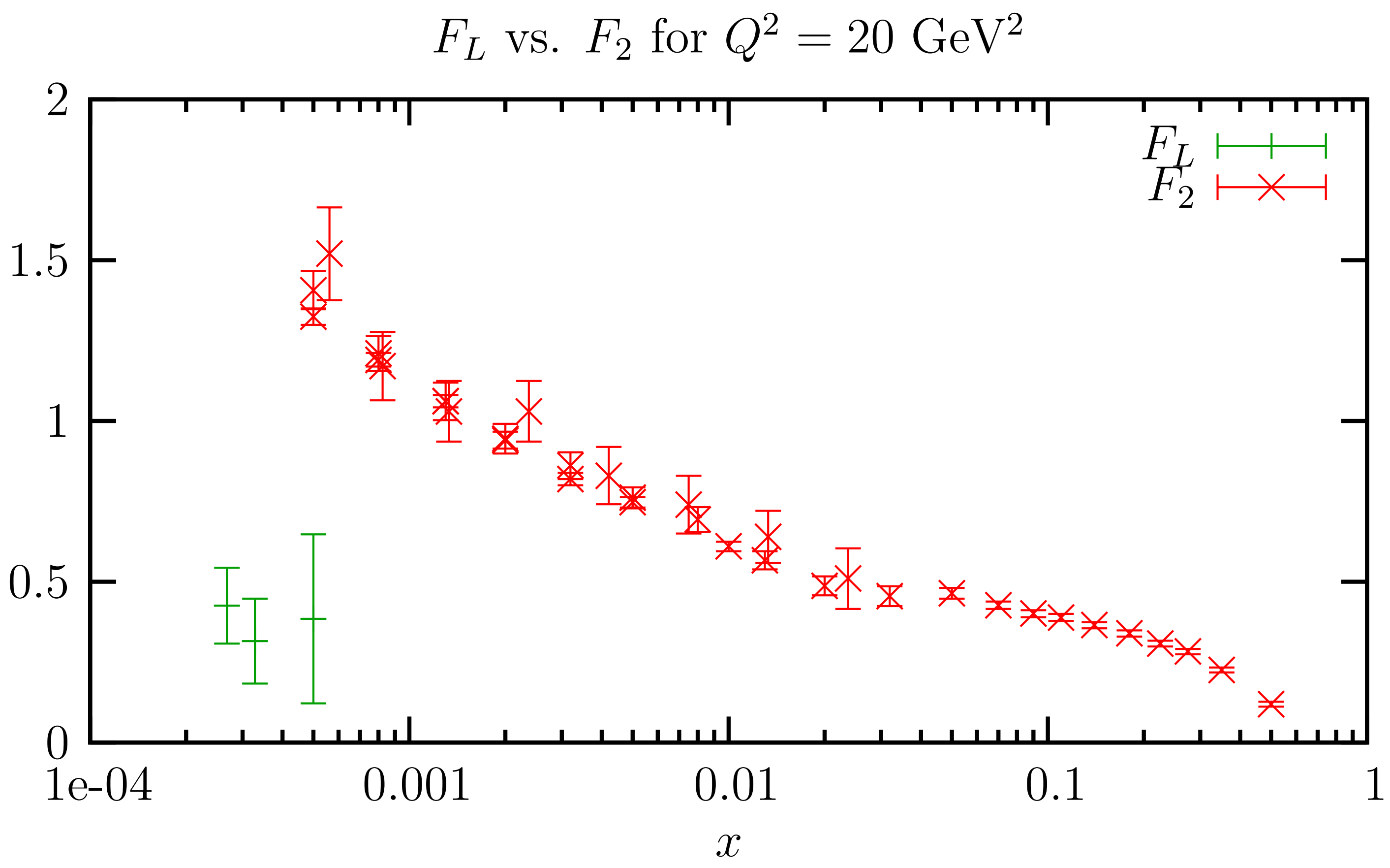
Bjorken scaling

- Bjorken scaling : F_2 depends very weakly on Q^2



Longitudinal F

- $F_L \equiv F_2 - 2xF_1$ is quite smaller than F_2 :



Analogy with the e- mu- cross-section

- In terms of F_1 and F_2, the DIS cross-section reads:

$$\frac{d\sigma_{e-N}}{dE'd\Omega} = \frac{\alpha_{\rm em}^2}{4m_{_N} E^2 \sin^4\frac{\theta}{2}}\left[2F_1 \sin^2\frac{\theta}{2} + \frac{m_{_N}^2}{\nu} F_2 \cos^2\frac{\theta}{2}\right]$$

- It is instructive to compare it to the $e^-\mu^-$ cross-section:

$$\frac{d\sigma_{e^-\mu^-}}{dE'd\Omega} = \frac{\alpha_{\rm em}^2 \delta(1-x)}{4m_{\mu} E^2 \sin^4\frac{\theta}{2}}\left[\sin^2\frac{\theta}{2} + \frac{m_{\mu}^2}{\nu} \cos^2\frac{\theta}{2}\right]$$

  - If the constituents of the nucleon that interact in the DIS process were spin $1/2$ point-like particles, we would have:

$$2F_1 = \frac{m_{_N}}{m_c}\delta(1-x_c) \quad , \quad F_2 = \frac{m_c}{m_{_N}}\delta(1-x_c)$$

    where m_c is some effective mass for the constituent (comparable to $m_{_N}$ because it is trapped inside the nucleon) and $x_c \equiv Q^2/2q\cdot p_c$ with p_c^μ the momentum of the constituent

- If $p_c^\mu = x_{_F} P^\mu$, then $x_c = x/x_{_F}$, and:

$$2F_1 \sim \delta(x - x_{_F}) \quad , \quad F_2 \sim \delta(x - x_{_F})$$

- The structure functions F_1 and F_2 would therefore not depend on Q^2, but only on x
- Conclusion : Bjorken scaling could be explained if the constituents of the nucleon that are probed in DIS are spin $1/2$ point-like particles

  The variable x measured in DIS would have to be identified with the fraction of momentum carried by the struck constituent

Naive parton model

- The historical parton model describes the nucleon as a collection of point-like fermions, called partons
- A parton of type i, carrying the fraction $x_{_F}$ of the nucleon momentum, gives the following contribution to the hadronic tensor :

$$4\pi W_i^{\mu\nu} = \int \frac{d^4 p'}{(2\pi)^4} 2\pi\delta(p'^2)(2\pi)^4\delta(x_{_F}P + q - p')$$

$$\times \Big\langle \big\langle x_{_F}P \big| J^{\mu\dagger}(0) \big| p' \big\rangle \big\langle p' \big| J^{\nu}(0) \big| x_{_F}P \big\rangle \Big\rangle_{\rm spin}$$

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$$\times \Big\langle \big\langle x_F P \big| J^{\mu\dagger}(0) \big| \boldsymbol{x_F P + q} \big\rangle \big\langle \boldsymbol{x_F P + q} \big| J^{\nu}(0) \big| x_F P \big\rangle \Big\rangle_{\text{spin}}$$

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$$\times \frac{e_i^2}{2}\,\mathrm{tr}\,(x_F \not{P}\gamma^\mu (x_F \not{P} + q)\gamma^\nu)$$

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$$\times\frac{e_i^2}{2}\,\mathrm{tr}\,(x_F \not{P}\gamma^\mu(x_F\not{P}+q)\gamma^\nu)$$

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$$\times \mathbf{2} e_i^2 \left(\boldsymbol{x}_F^2 \boldsymbol{P}^\mu \boldsymbol{P}^\nu + \boldsymbol{x}_F(\boldsymbol{P}^\mu \boldsymbol{q}^\nu + \boldsymbol{q}^\mu \boldsymbol{P}^\nu) - \boldsymbol{x}_F \boldsymbol{g}^{\mu\nu}\, \boldsymbol{P}\cdot\boldsymbol{q}\right)$$

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- If there are $f_i(x_F)dx_F$ partons of type i with a momentum fraction between x_F and $x_F + dx_F$, we have

$$W^{\mu\nu} = \sum_i \int_0^1 \frac{dx_F}{x_F}\, f_i(x_F)\, W_i^{\mu\nu}$$

- One obtains the following structure functions :

$$F_1 = \frac{1}{2}\sum_i e_i^2 f_i(x) \quad , \quad F_2 = 2xF_1$$

Naive parton model

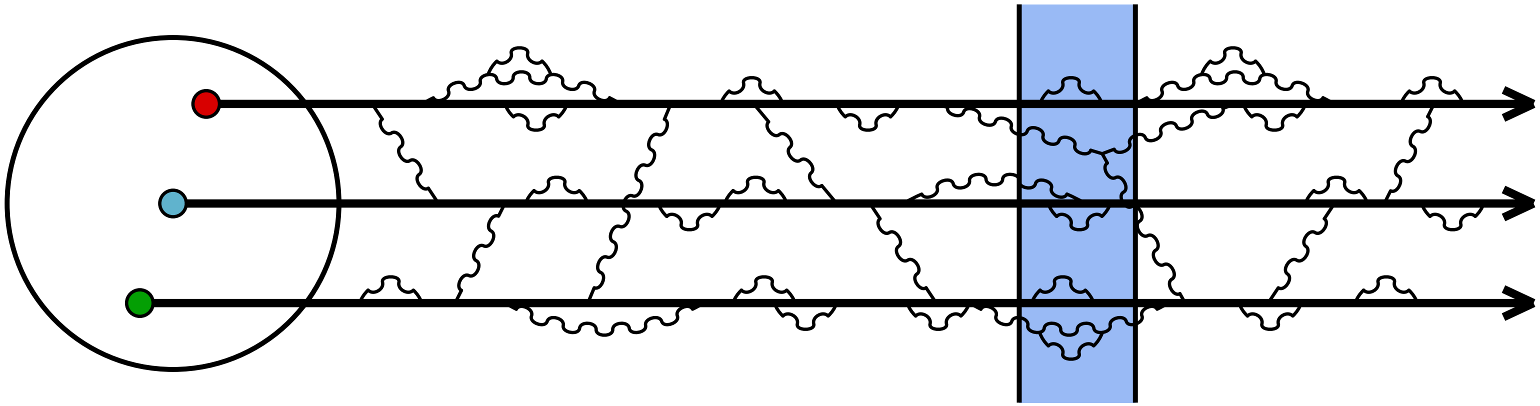
- This model provides an explicit realization of Bjorken scaling
- The relation $F_2 = 2xF_1$ implies that the cross-section between a longitudinally polarized photon and the nucleon is suppressed compared to that of a transverse photon
  - The observation of this property provides further support of the fact that the relevant constituents are spin $1/2$ fermions
  - If the partons were spin 0 particles, we would have

$$W_i^{\mu\nu} \propto (2x_{_F} P^\mu + q^\mu)(2x_{_F} P^\nu + q^\nu)$$

and it is easy to check that this leads to $F_1 = 0$ $(\sigma_{\rm transverse} = 0)$

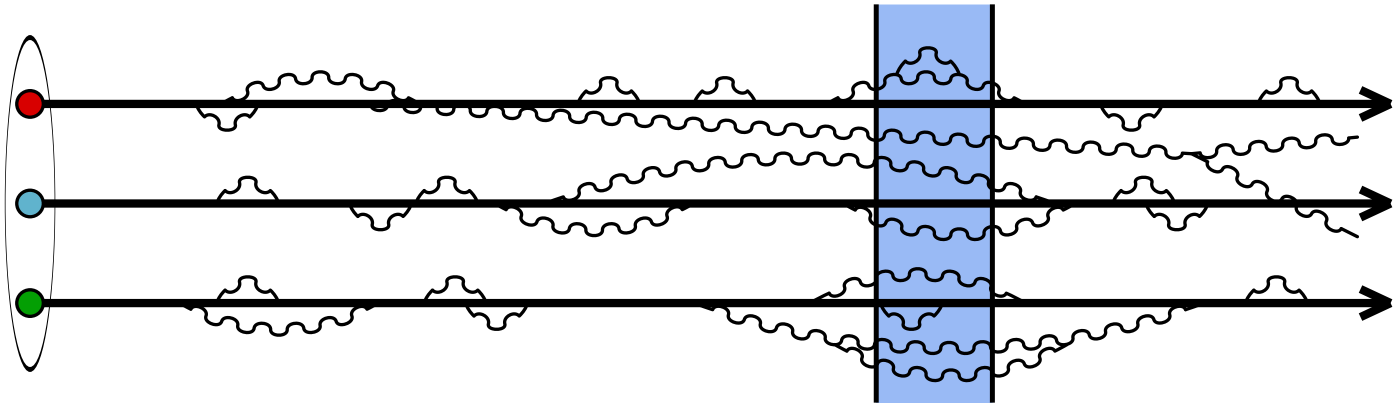
- Caveats and puzzles :
  - The parton model assumes that partons are free inside the nucleon. How can this be true in a strongly bound state ?
  - One would like to have a field theoretical description of what is going on, including the effect of interactions, quantum fluctuations, etc...

Field theory point of view



- A nucleon at rest is a very complicated object...
- Contains fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe

Field theory point of view



- Dilation of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe
  ▷ the constituents behave as if they were free
- Many fluctuations live long enough to be seen by the probe. The nucleon appears denser at high energy (it contains more gluons)

What would we learn ?

- The field theory that describes the interactions among partons should be able to explain the evolution with x of the parton distributions, since it comes from bremsstrahlung
- This field theory should also describe the evolution with Q^2 (i.e. the deviations from Bjorken scaling), which is due to the fact that the probe resolves more quantum fluctuations when Q^2 increases
- For the picture to be predictive, one should be able to prove from first principles the factorization of hadronic cross-section into a hard process (calculable?) and the parton distributions (not calculable?)

Kinematics of the Bjorken limit

- Bjorken limit : $Q^2, \nu \to +\infty \ , \quad x = \text{constant}$
- Go to a frame where the photon momentum is :

$$q^\mu = \frac{1}{m_N}(\nu, 0, 0, \sqrt{\nu^2 + m_N^2 Q^2})$$

- Therefore :

$$q^+ \equiv \frac{q^0 + q^3}{\sqrt{2}} \sim \frac{\nu}{m_N} \to +\infty$$

$$q^- \equiv \frac{q^0 - q^3}{\sqrt{2}} \sim m_N x \to \text{constant}$$

- Since $q \cdot y = q^+ y^- + q^- y^+ - \vec{\boldsymbol{q}}_\perp \cdot \vec{\boldsymbol{y}}_\perp$, the integration over y^μ is dominated by :

$$y^- \sim \frac{m_N}{\nu} \to 0 \quad , \quad y^+ \sim (m_N x)^{-1}$$

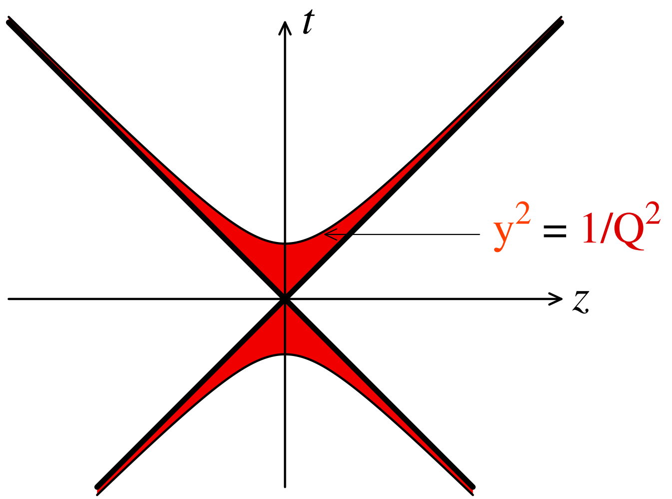
- Hence : $y^2 \leq 2y^+ y^- \sim 1/Q^2 \to 0$

Kinematics of the Bjorken limit

- $W_{\mu\nu}$ can be rewritten in terms of the commutator $[J^\dagger_\mu(y), J_\nu(0)]$. Thus, $y^2 \geq 0$ (causality). Therefore, the Bjorken limit is dominated by :

$$0 \leq y^2 \lesssim \frac{1}{Q^2} \to 0$$

i.e. by points very close to (and above) the light-cone



- Note : in this limit, the components of y^μ are not small...

Time ordered correlator of currents

- Consider a time-ordered product of currents :

$$4\pi T_{\mu\nu} \equiv i \int d^4y e^{iq\cdot y} \Big\langle \big\langle N(P) \big| T(J^\dagger_\mu(y) J_\nu(0)) \big| N(P) \big\rangle \Big\rangle_{\rm spin}$$

- At fixed Q^2, the functions $T_{1,2}(\nu, Q^2)$ are analytic in ν with cuts on the real axis starting at $\pm Q^2/2$
- Like $W_{\mu\nu}$, $T_{\mu\nu}$ has a tensor decomposition, with structure functions T_1 and T_2 :

$$T_{\mu\nu} = -T_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{T_2}{P\cdot q}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right)$$

- F_r is related to the discontinuity of T_r across the cut $(W_{\mu\nu} = 2\,{\rm Im}\,T_{\mu\nu})$

Operator Product Expansion

- Consider the correlator $\langle \mathcal{A}(0)\mathcal{B}(x)\phi(x_1)\cdots\phi(x_n)\rangle$ where $\mathcal{A}$ and $\mathcal{B}$ are two local operators, possibly composite
- When $|x| \to 0$, this function is usually singular because products of operators at the same point are ill-defined
- These singularities do not depend on the nature and localization of the extra fields $\phi(x_i)$
- One can obtain them from an expansion of the form

$$\mathcal{A}(0)\mathcal{B}(x) \underset{|x|\to 0}{=} \sum_i C_i(x)\ \mathcal{O}_i(0)$$

  - the $\mathcal{O}_i(0)$ are local operators with the quantum numbers of $\mathcal{A}\mathcal{B}$
  - the $C_i(x)$ are numbers that contain the singular behavior
- When $|x| \to 0$, $C_i(x)$ behaves as

$$C_i(x) \underset{|x|\to 0}{\sim} |x|^{\mathrm{d}(\mathcal{O}_i)-\mathrm{d}(\mathcal{A})-\mathrm{d}(\mathcal{B})} \qquad \text{(up to logs)}$$

▷ only the operators with a low mass dimension matter

Operator Product Expansion of T(JJ)

- The local operators that may appear in the OPE of $T(J_\mu^\dagger(y)J_\nu(0))$ can be classified according to the representation of the Lorentz group to which they belong. Let us denote them $\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}$ where s is the "spin" of the operator, and the index i labels the various operators having the same tensor structure. The OPE of T_1 and T_2 has the form :

$$\sum_{s,i} C^{s,i}_{\mu_1\cdots\mu_s}(y)\ \mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}(0)$$

- The Wilson coefficients of these operators must have the following structure :

$$C^{s,i}_{\mu_1\cdots\mu_s}(y) \equiv y_{\mu_1}\cdots y_{\mu_s}\ C_{s,i}(y^2)$$

- The expectation values in the nucleon state are of the form :

$$\Big\langle\big\langle N(P)\big|\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}(0)\big|N(P)\big\rangle\Big\rangle_{\rm spin} = \left[P^{\mu_1}\cdots P^{\mu_s} + \text{trace terms}\right]\left\langle\mathcal{O}_{s,i}\right\rangle$$

Power counting and ‘twist’

- Let $\boldsymbol{d}_{s,i}$ be the mass dimension of the operator $\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}$
- Then, the dimension of $C_{s,i}(y^2)$ is $6+s-\boldsymbol{d}_{s,i}$
  ▷ this function scales as $(y^2)^{(\boldsymbol{d}_{s,i}-s-6)/2}$ (up to logs)
- In a standard OPE, where $y_\mu\to 0$, the factor $y_{\mu_1}\cdots y_{\mu_n}$ would bring an extra $|y|^s$ to this scaling behavior, making the coefficient of $\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}$ scale as $|y|^{\boldsymbol{d}_{s,i}-6}$, and high-dimension operators would be suppressed
- But in the Bjorken limit, the components of y_μ do not go to zero, and therefore the factor $y_{\mu_1}\cdots y_{\mu_n}$ should not be counted. In this case, it is the difference $\boldsymbol{d}_{s,i}-s$ (called the “twist”) that controls the scaling behavior of the coefficient
- The leading behavior of $T(J_\mu^\dagger(y)J_\nu(0))$ is controlled by the operators having the smallest twist. There is an infinity of them, because the dimension $\boldsymbol{d}_{s,i}$ can be compensated by a higher spin

Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_{s,i} \langle \mathcal{O}_{s,i} \rangle \int d^4y \; e^{iq\cdot y} \; C_{s,i}(y^2) \; (P\cdot y)^s$$

Operator Product Expansion of T(JJ)

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Operator Product Expansion of T(JJ)

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$$\sum_{s,i} \langle \mathcal{O}_{s,i} \rangle \left(-\boldsymbol{i} P_\mu \frac{\boldsymbol{\partial}}{\boldsymbol{\partial q_\mu}} \right)^s \int \boldsymbol{d^4 y}\, e^{\boldsymbol{iq\cdot y}}\, C_{s,i}(y^2)$$

Operator Product Expansion of T(JJ)

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$$\sum_{s,i} \left\langle \mathcal{O}_{s,i} \right\rangle \left(-iP_\mu \frac{\partial}{\partial q_\mu} \right)^s \widetilde{\boldsymbol{C}}_{s,i}(-\boldsymbol{q_\mu q^\mu})$$

Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_{s,i} \langle \mathcal{O}_{s,i} \rangle \left(-iP_\mu \frac{\boldsymbol{\partial}}{\boldsymbol{\partial q_\mu}} \right)^s \widetilde{C}_{s,i}(\boldsymbol{-q_\mu q^\mu})$$

Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_{s,i} \left\langle \mathcal{O}_{s,i} \right\rangle \left(-\boldsymbol{2} iP \cdot \boldsymbol{q}\right)^s \; \widetilde{C}_{s,i}^{(\boldsymbol{s})}(-\boldsymbol{q_\mu q^\mu})$$

Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_{s} \boldsymbol{x^{-s}} \sum_{i} \left\langle \mathcal{O}_{s,i} \right\rangle \underbrace{(-i)^s\, \boldsymbol{Q^{2s}}\, \widetilde{C}^{(s)}_{s,i}(Q^2)}_{D_{s,i}(Q^2)}$$

Operator Product Expansion of T(JJ)

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- Note: from their definitions, T_1 and T_2 differ by a power of P. Having the same dimension, they differ in fact by a factor x :

$$T_1(x,Q^2) = \sum_s x^{-s} \sum_i \left\langle \mathcal{O}_{s,i} \right\rangle \, D_{1;s,i}(Q^2)$$

$$T_2(x,Q^2) = \sum_s x^{1-s} \sum_i \left\langle \mathcal{O}_{s,i} \right\rangle \, D_{2;s,i}(Q^2)$$

  - Since all the powers of x and Q^2 have been counted explicitly, $D_{1;s,i}$ and $D_{2;s,i}$ can only differ by constant factors and logs

Operator Product Expansion of T(JJ)

- The coefficient function $C_{s,i}(y^2)$ behaves like $y^{d_{s,i}-s-6}$
  - Its Fourier transform $\widetilde{C}_{s,i}(Q^2)$ scales as $Q^{2+s-d_{s,i}}$
  - So does $D_{r;s,i}(Q^2) \propto Q^{2s}\widetilde{C}^{(s)}_{s,i}(Q^2)$
- Therefore, if the leading twist operators correspond to $d_{s,i} - s = 2$, we have Bjorken scaling automatically
- The coefficients $D_{r;s,i}(Q^2)$ are calculable in perturbation theory, and do not depend on the target
- The matrix elements $\langle \mathcal{O}_{s,i} \rangle$ are non perturbative, and contain all the information about the target
- At this stage, the predictive power of this approach is limited to scaling properties, because we do not know the target dependent factors $\langle \mathcal{O}_{s,i} \rangle$
  However, when we bring the renormalization group machinery into the game, we will also predict deviations from these scaling laws

Moments of F1 and F2

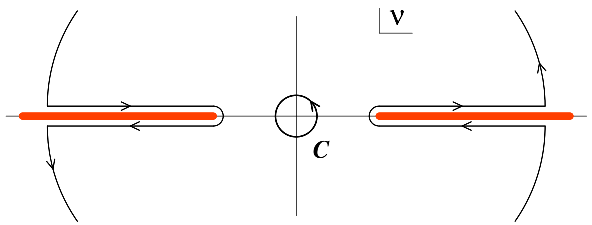
- The OPE provides a Taylor expansion of $T_{1,2}$ in powers of x^{-1} (all the x dependence is in the factor x^{-s}) :

$$T_r = \sum_s t_r(s,Q^2)\; x^{a_r - s} = \sum_s t_r(s,Q^2)\; \left(\frac{2}{Q^2}\right)^2 \; \nu^{s-a_r}$$

with $a_1 = 0, a_2 = 1$. From this, we get :

$$t_r(s,Q^2) = \frac{1}{2\pi i}\left(\frac{Q^2}{2}\right)^{s-a_r} \int_{\mathcal{C}} \frac{d\nu}{\nu}\nu^{a_r - s}\; T_r(\nu,Q^2)$$

- Do the integration by wrapping the contour around the cuts, and use the relation between F_r and the discontinuity of T_r accros the cut :



$$t_r(s,Q^2) = \frac{2}{\pi}\int_0^1 \frac{dx}{x}\; x^{s-a_r}\; F_r(x,Q^2)$$

▷ the OPE gives the moments of the DIS structure functions

Bare Wilson coefficients

- Now, let us assume that the underlying field theory of strong interactions has spin $1/2$ fermions (quarks) and vector bosons (gluons). The operators with the lowest twist are (dimension $s+2$ and spin s, hence twist 2) :

$$\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s} \equiv \overline{\psi}_f \gamma^{\{\mu_1}\partial^{\mu_2}\cdots\partial^{\mu_s\}}\psi_f$$
$$\mathcal{O}_{s,g}^{\mu_1\cdots\mu_s} \equiv F_\alpha{}^{\{\mu_1}\partial^{\mu_2}\cdots\partial^{\mu_{s-1}}F^{\mu_s\}\alpha}$$

  where the brakets $\{\cdots\}$ denote a symmetrization of the indices $\mu_1\cdots\mu_s$ and a subtraction of the trace terms on those indices

- In order to compute the Wilson coefficients, one can exploit the fact that they do not depend on the target:

  consider an elementary target (single fermion or vector boson) for which everything is calculable (including the $\langle\mathcal{O}_{s,i}\rangle$, that are non perturbative if the target is a nucleon)

Bare Wilson coefficients

- Consider a quark state of a given flavor and given spin. At lowest order, one has :

$$\big\langle f,\sigma\big|\mathcal{O}^{\mu_1\cdots\mu_s}_{s,f'}\big|f,\sigma\big\rangle = \delta_{ff'}\overline{u}_\sigma(P)\gamma^{\{\mu_1}u_\sigma(P)P^{\mu_2}\cdots P^{\mu_s\}}$$
$$\big\langle f,\sigma\big|\mathcal{O}^{\mu_1\cdots\mu_s}_{s,g}\big|f,\sigma\big\rangle = 0$$

- Averaging over the spin of the quark, and comparing with $P^{\mu_1}\cdots P^{\mu_s}\big\langle\mathcal{O}_{s,i}\big\rangle$, leads to :

$$\big\langle\mathcal{O}_{s,f'}\big\rangle_f = \delta_{ff'}\quad,\qquad \big\langle\mathcal{O}_{s,g}\big\rangle_f = 0$$

- On the other hand, one can calculate directly the expectation value of the current-current correlator in this quark state. This is simply done by taking the parton model results for $F_{1,2}$ and using dispersion relations to get $T_{1,2}$:

$$t_1(s,Q^2) = \frac{1}{\pi}\,e_f^2\quad,\qquad t_2(s,Q^2) = \frac{2}{\pi}\,e_f^2$$

Bare Wilson coefficients

- Therefore, the bare coefficient functions are :

$$D_{1;s,f}(Q^2) = \frac{1}{\pi}\, e_f^2 \quad , \qquad D_{2;s,f}(Q^2) = \frac{2}{\pi}\, e_f^2$$

- Repeating the same steps with a vector boson state gives :

$$D_{1;s,g}(Q^2) = D_{2;s,g}(Q^2) = 0$$

if the vector bosons are assumed to be electrically neutral

- Going back to a nucleon target, it is convenient to define parton distribution functions as the $f_i(x)$ whose moments are :

$$\int_0^1 \frac{dx}{x}\, x^s\, f_i(x) = \langle \mathcal{O}_{s,i} \rangle$$

so that :

$$F_1(x) = \frac{1}{2}\sum_f e_f^2 f_f(x) \quad , \qquad F_2(x) = x\sum_f e_f^2 f_f(x) = 2x F_1(x)$$

Learnings from free field theory

- Despite the fact that the result is embarrassingly similar to what we obtained in a much simpler way in the naive parton model, this exercise has taught us several things :
- Bjorken scaling can be derived from first principles in a field theory of free fermions (somewhat disturbing given that these fermions are constituents of a strongly bound state)
- We now have an operatorial definition of the distribution $f_i(x)$ (not calculable perturbatively however)
- More importantly, the experimental observation of Bjorken scaling is telling us that the field theory of strong interactions must become a free theory in the limit $Q^2 \to +\infty$
  ▷ asymptotic freedom
- As shown by Gross, Wilczek, Politzer in 1973, non-abelian gauge theories with a reasonable number of fermionic fields (like QCD with 6 flavors of quarks) have this property

Operator rescaling

- In the previous discussion, we have implicitly assumed that there is no scale dependence in the moments $\langle \mathcal{O}_{s,i} \rangle$ of the distribution functions
- In fact, they depend on the renormalization scale μ^2, so that the distribution functions are scale dependent as well
- Of course, the structure functions F_1 and F_2, being observable quantities, cannot depend on the renormalization scale μ^2. This means that there should also be a μ^2 dependence in the coefficient functions, in order to compensate the μ^2 dependence from $\langle \mathcal{O}_{s,i} \rangle$
- The Wilson coefficients will be some trivial power of Q^2 imposed by their dimension (that alone would imply Bjorken scaling), times a function of the ratio Q^2/μ^2. This corrective factor will violate Bjorken scaling

Callan-Symanzik equation

- Consider the following correlators :

$$G_{JJ}(x) \equiv \langle T(J(x)J(0))\rangle \quad , \qquad G_{s,i}(0) \equiv \langle \mathcal{O}_{s,i}(0)\rangle$$

$$G_{JJ}(x) = \sum_{s,i} C_{s,i}(x) G_{s,i}(0)$$

- The Callan-Symanzik equations for G_{JJ} and $G_{s,i}$ are :

$$\left[\mu\partial_\mu + \beta\partial_g + 2\gamma_J\right] G_{JJ} = 0$$

$$\left[(\mu\partial_\mu + \beta\partial_g)\,\delta_{ij} + \gamma_{s,ij}\right] G_{s,j} = 0$$

where β is the beta function, γ_J the anomalous dimension of the current J (in fact $\gamma_J = 0$ for conserved currents), and $\gamma_{s,ij}$ the matrix of anomalous dimensions for the $\mathcal{O}_{s,i}$ (the operator mixing is limited to operators with the same Lorentz structure)

- By combining the previous equations, one gets :

$$\left[(\mu\partial_\mu + \beta\partial_g)\,\delta_{ij} - \gamma_{s;ji}\right] C_{s,j} = 0$$

Solution of the CS equation

- The dimensionless coefficients $D_{r;s,i}(Q,\mu,g)$ are in fact functions $D_{r;s,i}(Q/\mu,g)$. Under rescalings of Q, they obey :

$$\left[\left(-Q\partial_Q+\beta(g)\partial_g\right)\delta_{ij}-\gamma_{s,ji}(g)\right]D_{r;s,j}(Q/\mu,g)=0$$

- In order to solve this equation, let us first introduce the running coupling $\overline{g}(Q,g)$ such that :

$$\ln(Q/Q_0)=\int_g^{\overline{g}(Q,g)}\frac{dg'}{\beta(g')}$$

(this is equivalent to $Q\partial_Q\overline{g}(Q,g)=\beta(\overline{g}(Q,g))$ and $\overline{g}(Q_0,g)=g$)

- Any function $F(\overline{g}(Q,g))$ obeys

$$\left[-Q\partial_Q+\beta(g)\partial_g\right]F=0$$

- We also have

$$\left[-Q\partial_Q+\beta(g)\partial_g\right]e^{-\int_{Q_0}^{Q}\frac{dM}{M}\gamma(\overline{g}(M,g))}=\left[e^{-\int_{Q_0}^{Q}\frac{dM}{M}\gamma(\overline{g}(M,g))}\right]\gamma(g)$$

Solution of the CS equation

- Therefore, the Wilson coefficients at scale Q can be expressed in terms of the Wilson coefficients at scale Q_0 by :

$$D_{r;s,i}(Q/\mu, g) = D_{r;s,j}(Q_0/\mu, \overline{g}(Q,g)) \left[e^{-\int_{Q_0}^{Q} \frac{dM}{M} \gamma_s(\overline{g}(M,g))} \right]_{ji}$$

- If the underlying theory is asymptotically free, like QCD, then at large Q the coupling is small and we can approximate :

$$\gamma_{s,ij}(\overline{g}) = \overline{g}^2 A_{ij}(s) \quad , \qquad \overline{g}^2(Q,g) = \frac{8\pi^2}{\beta_0 \ln(Q/\Lambda_{QCD})}$$

where the $A_{ij}(s)$ are given by a 1-loop perturbative calculation

- Finally, the solution can be rewritten as :

$$D_{r;s,i}(Q/\mu, g) = D_{r;s,j}(Q_0/\mu, \overline{g}(Q,g)) \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})} \right)^{-\frac{8\pi^2}{\beta_0} A(s)} \right]_{ji}$$

Scaling violations in F1 and F2

- The moments of the structure function F_1 at scale Q^2 read :

$$\int_0^1 \frac{dx}{x}\, x^s\, F_1(x,Q^2) = \sum_{i,f} \frac{e_f^2}{2} \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})} \right)^{-\frac{8\pi^2}{\beta_0} A(s)} \right]_{fi} \langle \mathcal{O}_{s,i} \rangle_{Q_0}$$

- F_1 takes the parton model form $F_1(x,Q^2) = \frac{1}{2}\sum_f e_f^2\, f_f$, provided we define quark distributions from their moments:

$$\int_0^1 \frac{dx}{x}\, x^s\, f_f(x,Q^2) \equiv \sum_{i} \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})} \right)^{-\frac{8\pi^2}{\beta_0} A(s)} \right]_{fi} \langle \mathcal{O}_{s,i} \rangle_{Q_0}$$

  - The quark distribution is now Q^2 dependent
  - It depends on the expectation value of operators involving gluons

- Scaling violations at LO preserve the Callan-Gross relation at large Q :

$$F_2(x,Q^2) = 2x F_1(x,Q^2)$$

Probabilistic interpretation

- In order to make the interpretation of the Q dependence more transparent, let us introduce as well a gluon distribution, even though it is not probed directly in DIS :

$$\int_0^1 \frac{dx}{x}\, x^s\, f_g(x,Q^2) \equiv \sum_i \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})}\right)^{-\frac{8\pi^2}{\beta_0}A(s)}\right]_{gi} \left\langle \mathcal{O}_{s,i}\right\rangle_{Q_0}$$

- The derivative of the moments of the parton distributions with respect to $\ln(Q^2)$ is :

$$Q^2\frac{\partial f_i(s,Q^2)}{\partial Q^2} = -\frac{\overline{g}^2(Q,g)}{2} A_{ji}(s) f_j(s,Q^2)$$

- In order to go further, we need the following result :

$$A(s)f(s) = \int_0^1 \frac{dx}{x}\, x^s \int_x^1 \frac{dy}{y}\, A(x/y) f(y)$$

Probabilistic interpretation

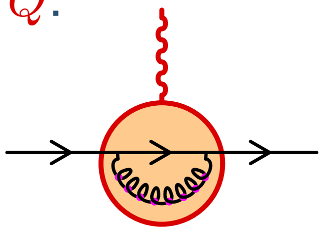
- Define the splitting functions P_{ij} from their moments :

$$\int_0^1 \frac{dx}{x}\, x^s\, P_{ij}(x) \equiv -4\pi^2 A_{ij}(s)$$

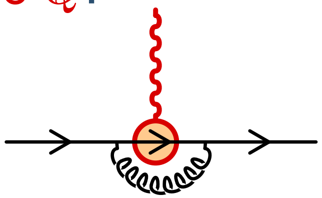
- Therefore, one has the following evolution equation for $f_i(x,Q^2)$ (DGLAP) :

$$Q^2 \frac{\partial f_i(x,Q^2)}{\partial Q^2} = \frac{\overline{g}^2(Q,g)}{8\pi^2} \int_x^1 \frac{dy}{y}\, P_{ji}(x/y) f_j(y,Q^2)$$

- Interpretation : the resolution of the γ^* changes with Q
  - Low Q:



  - Large Q:



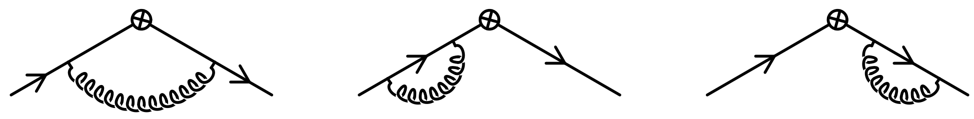
  - $\overline{g}^2 P_{ji}(z)$ describes the splitting $j \to i$, where the daughter parton takes the fraction z of the momentum of the original parton

Anomalous dimensions

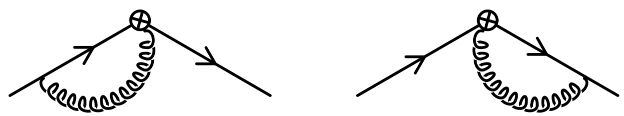
- The anomalous dimension of an operator $\mathcal{O}$ is given by :

$$\gamma_{\mathcal{O}} = \frac{\mu}{Z_{\mathcal{O}}}\frac{\partial Z_{\mathcal{O}}}{\partial \mu} \quad , \qquad \text{where } \mathcal{O}_{\text{renormalized}} = Z_{\mathcal{O}}{}^{-1}\mathcal{O}_{\text{bare}}$$

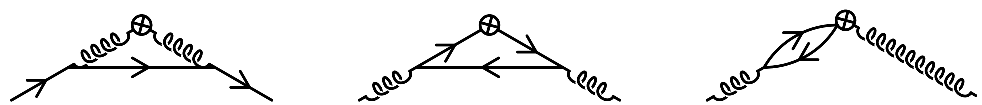
- At 1-loop, the operator $\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s}$ has the following corrections :



- Moreover, to ensure gauge invariance, the operator $\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s}$ should be defined as : $\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s} \equiv \overline{\psi}_f \gamma^{\{\mu_1} D^{\mu_2} \cdots D^{\mu_s\}}\psi_f$
Therefore, one has also the following 1-loop diagrams :



- Finally, there are some diagrams mixing $\mathcal{O}_{s,f}$ and $\mathcal{O}_{s,g}$



Anomalous dimensions

- At 1-loop, the coefficients $A_{ij}(s)$ in the anomalous dimensions are :

$$A_{gg}(s) = \frac{1}{2\pi^2}\left\{3\left[\frac{1}{12} - \frac{1}{s(s-1)} - \frac{1}{(s+1)(s+2)} + \sum_{j=2}^{s}\frac{1}{j}\right] + \frac{N_f}{6}\right\}$$

$$A_{fg}(s) = \frac{1}{2\pi^2}\left\{\frac{1}{s+2} + \frac{2}{s(s+1)(s+2)}\right\}$$

$$A_{gf}(s) = \frac{3}{8\pi^2}\left\{\frac{1}{s+1} + \frac{2}{s(s-1)}\right\}$$

$$A_{ff'}(s) = \frac{3}{8\pi^2}\left\{1 - \frac{2}{s(s+1)} + 4\sum_{j=2}^{s}\frac{1}{j}\right\}\delta_{ff'}$$

- All the non-singlet linear combinations: $\sum_f a_f \mathcal{O}_{s,f}$ with $\sum_f a_f = 0$ are eigenvectors of the matrix of anomalous dimensions, with an eigenvalue $A_{ff}(s)$
  These linear combinations do not mix with the remaining two operators, $\sum_f \mathcal{O}_{s,f}$ and $\mathcal{O}_{s,g}$, through renormalization

Valence sum rules (s=1)

- In the case of $s = 1$, the anomalous dimension of the non-singlet quark operators is

$$A_{ff}(s = 1) = 0$$

- Going back to the evolution equation for the moments of quark distributions, this means that we have :

$$\frac{\partial}{\partial Q^2}\left\{\int_0^1 dx \sum_f a_f f_f(x, Q^2)\right\} = 0$$

for any linear combination such that $\sum_f a_f = 0$

- For instance, for a nucleon, this implies that the number of u quarks minus the number of d quarks is independent of Q^2

- Interpretation : the production of extra quarks by $g \to q\bar{q}$ produces quarks of all flavors in equal numbers

Momentum sum rule (s=2)

- In the singlet sector, the matrix of anomalous dimensions for $s = 2$ reads :

$$A_{\rm singlet}(s=2) = \frac{1}{3\pi^2}\begin{pmatrix} \frac{N_f}{4} & \frac{2N_f}{3} \\ \frac{1}{2} & \frac{4}{3} \end{pmatrix}$$

- This matrix has a vanishing determinant, which means that a linear combination of the flavor singlet operators is not renormalized : $8\mathcal{O}^{\mu\nu}_{2,g} - 3\sum_f \mathcal{O}^{\mu\nu}_{2,f}$

- This leads also to a sum rule :

$$\frac{\partial}{\partial Q^2}\left\{\int_0^1 dx\, x\left[3\sum_f f_f(x,Q^2) - 8 f_g(x,Q^2)\right]\right\} = 0$$

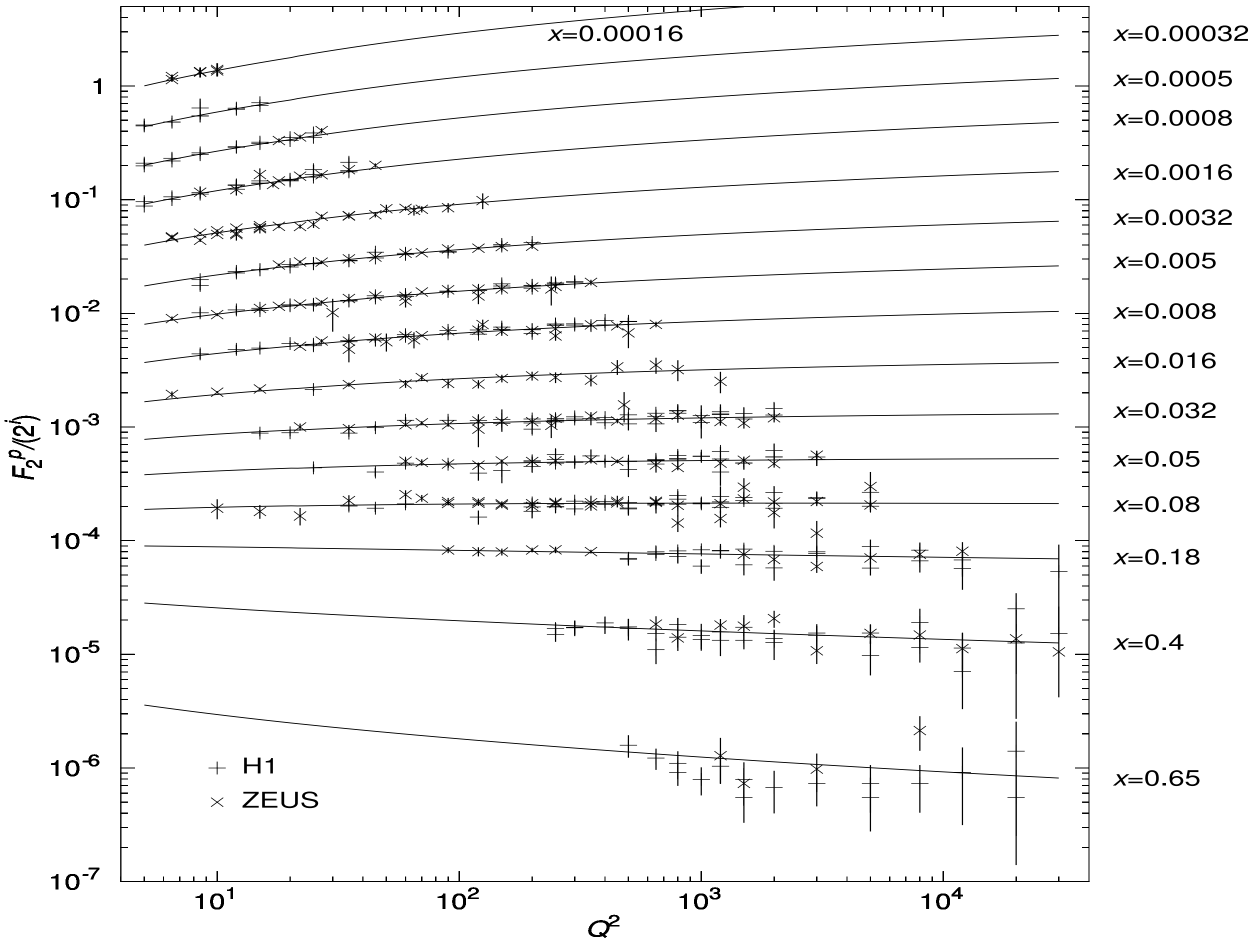
- Interpretation : the total longitudinal momentum of the target is conserved, and the momentum that goes into the newly produced gluons must be taken from the quarks

Practical strategy

- Due to the non-perturbative nature of the parton distributions at a given fixed scale Q, it does not make sense to try to predict the value of F_r at a given Q out of nothing
- Instead,
  - fit the parton distributions from the measurement of F_r at a moderately low scale Q_0
  - using DGLAP, evolve them to a higher scale Q
  - predict the values of the structure functions F_r at the scale Q
  - compare with DIS measurements
- This approach can be systematically improved by going to higher order, both for the hard subprocess, and for the splitting functions and beta function
- Current state of the art :
  - NLO program fully implemented
  - NNLO splitting functions and beta function are known

HERA results for F2

- HERA results and NLO DGLAP fit :



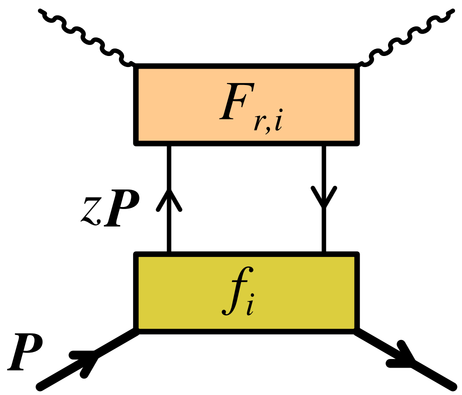
Factorization in DIS

- The DIS structure functions can be written as :

$$F_r(x, Q^2) = \sum_i \int_x^1 dz\, f_i(z, Q^2) F_{r,i}(x/z, Q^2) + \mathcal{O}\left(\frac{m_N^2}{Q^2}\right)$$

  - $F_{r,i}$ is the structure function for a target parton i (at leading order, it is non-zero only for quarks)
  - x/z is the Bjorken-x variable for the system $\gamma^* i$

- Schematically, one can represent this factorization as :

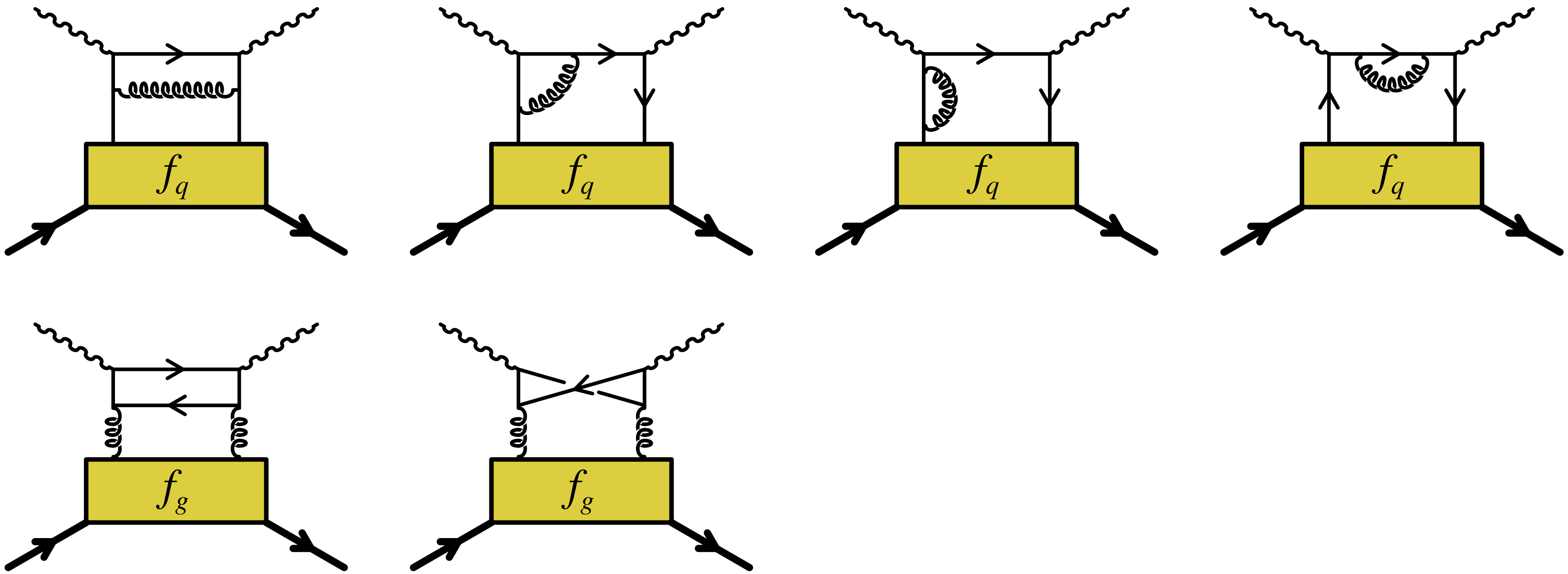


Factorization in DIS

- In perturbation theory, the terms included by the RG evolution correspond to factors of g^2 enhanced by large logarithms :

$$g^2 \ln\left(Q^2/\mu^2\right) \qquad \text{where } \mu^2 \text{ is some soft cutoff}$$

- The logs are due to collinear divergences in loop corrections to $F_{r,i}$. The first power of $g^2\ln(Q^2/\mu^2)$ comes from :

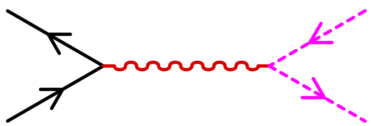


Factorization in DIS - Beyond LO

- For DIS, the procedure for going to NLO is straightforward and dictated by the OPE approach. One needs the following quantities at NLO :
  - coefficient functions
  - beta function
  - anomalous dimensions (or splitting functions)
- Changes compared to LO :
  - The Callan-Gross relation does not hold anymore
  - There are various ways to define parton distributions: they are not directly measurable, and one should regard them as an intermediate device to relate various measurable cross-sections. The hard scattering part of the factorization formula must be changed accordingly
  - Some parton sum rules may get modified at NLO

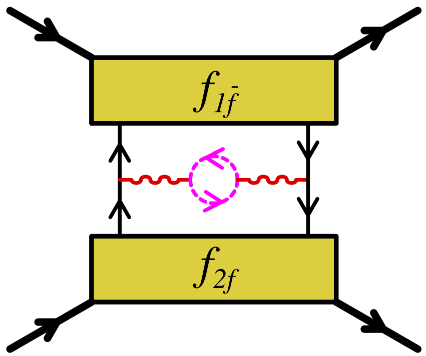
Factorization in Drell-Yan

- The Drell-Yan process is a reaction between two hadrons in which a virtual photon is produced, that later decays into a lepton-antilepton pair
- At the parton level, the simplest process responsible for this reaction is a $q\bar{q} \to \gamma^*$ annihilation :



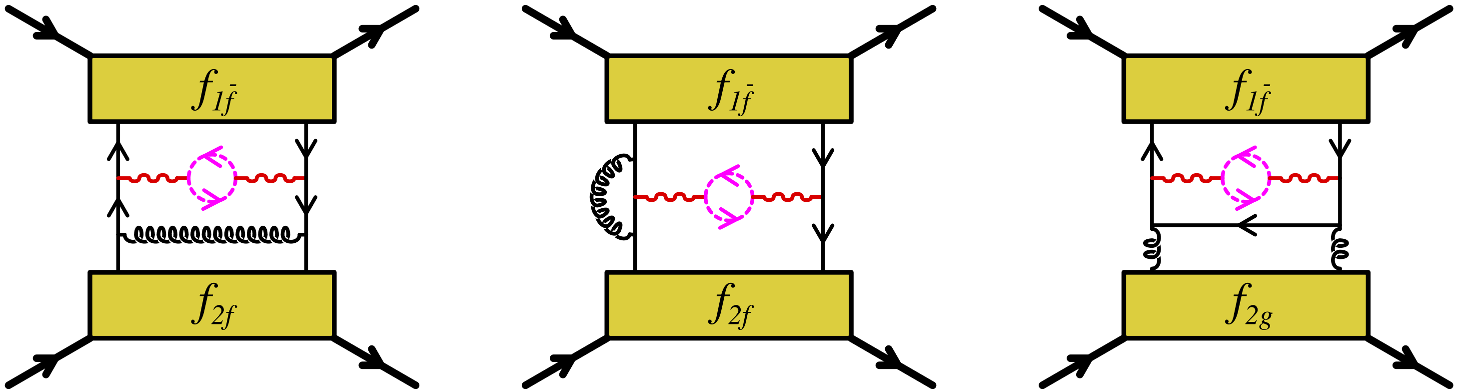
- The cross-section in the naive parton model reads :

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9Q^4}\sum_f e_f^2 \int_0^1 dx_1\, dx_2\, x_1 x_2\, \delta(x_1 x_2 - Q^2/s)$$
$$\times\Big[f_{1f}(x_1) f_{2\bar{f}}(x_2) + f_{1\bar{f}}(x_1) f_{2f}(x_2)\Big]$$



Factorization in Drell-Yan

- Sample of loop diagrams with leading-log contributions :



- At LO, the naive parton model Drell-Yan formula remains true after resummation of all the leading log corrections, modulo the replacement $f_{if}(x_i) \to f_{if}(x_i, Q^2)$, with the same distribution functions as in DIS :

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9Q^4} \sum_f e_f^2 \int_0^1 dx_1\, dx_2\, x_1 x_2\, \delta(x_1 x_2 - Q^2/s)$$
$$\times \Big[f_{1f}(x_1, Q^2) f_{2\bar{f}}(x_2, Q^2) + f_{1\bar{f}}(x_1, Q^2) f_{2f}(x_2, Q^2) \Big]$$

Collinear factorization

- Factorization is the possibility to resum all the powers $[g^2 \ln(Q^2/\mu^2)]^n$ into universal parton distributions
  - The neglected contributions are suppressed by powers of $1/Q$
  - The hard subprocess is infrared safe
- The "bare" parton distributions are turned into Q-dependent distributions, that obey the DGLAP equation
- The universality of the parton distributions confers to QCD a much stronger predictive power, since the distributions measured in DIS can be used to predict other processes
- Interactions due to soft gluons in the final state cancel when one sums over degenerate final states (KLN)
- Crucial for factorization is the large difference between the short and long timescales : at high energy, internal hadronic timescales get dilated while the duration of the interaction goes to zero because of Lorentz contraction

Separation of timescales

- Consider a massless parton of longitudinal momentum p splitting into two partons of longitudinal momenta zp and $(1-z)p$ and transverse momenta $+\vec{\boldsymbol{k}}_\perp$ and $-\vec{\boldsymbol{k}}_\perp$. Their energies are :

$$E_0 = p \quad , \quad E_1 \approx |z|p + \frac{\vec{\boldsymbol{k}}_\perp^2}{2|z|p} \quad , \quad E_2 \approx |1-z|p + \frac{\vec{\boldsymbol{k}}_\perp^2}{2|1-z|p}$$

- The lifetime of this fluctuation is given by :

$$\tau_{\rm fluct}^{-1} \sim E_1 + E_2 - E_0 = (|z| + |1-z| - 1)p + \frac{\vec{\boldsymbol{k}}_\perp^2}{2p}\left(\frac{1}{|z|} + \frac{1}{|1-z|}\right)$$

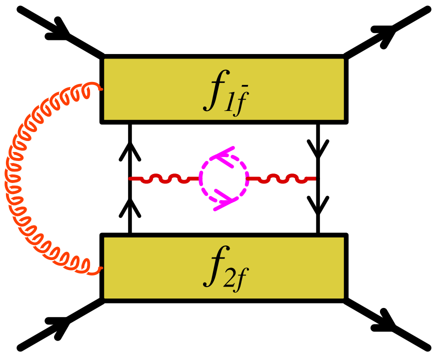
- If $z < 0$ or $z > 1$, this fluctuation is very short-lived
- If $0 < z < 1$, $|z| + |1-z| = 1$, and the lifetime becomes :

$$\tau_{\rm fluct} \sim 2z(1-z)p/\vec{\boldsymbol{k}}_\perp^2$$

- This must be compared with the interaction time of the virtual photon : $\tau_{\rm int} \sim p/Q^2$. For the collinear contributions: $\vec{\boldsymbol{k}}_\perp^2 \ll Q^2$, hence $\tau_{\rm int} \ll \tau_{\rm fluct}$

Initial state interactions

- A major complication in processes with two incoming hadrons, like Drell-Yan, is the possibility that the two hadrons may be connected by soft gluons before the collision :



- This could have the disastrous effect of making the parton distributions of a hadron non-universal
- Such interactions can be seen as the interactions of one projectile with the Coulomb field of the other projectile
- For very high energy projectiles, Lorentz contraction implies that the field strength $F_{\mu\nu}$ is localized on a sheet perpendicular to the trajectory. Therefore, it cannot affect the contents of the other hadron before the collision

Infrared safe final states

- Infrared divergences cancel when one sums over all the possible final states (Kinoshita-Lee-Nauenberg theorem)
- One can see such a cross-section as the sum of cuts through a forward scattering amplitude. Each individual cut is a divergent contribution, but the sum of all the cuts is finite
- Completely inclusive final states are not the only ones to be infrared safe. Consider the following weighted cross-section :

$$\sigma_S \equiv \int [d\Phi_n] \frac{d\sigma}{d\Phi_n} S_n(p_1, \cdots, p_n)$$

  - Such a final state is infrared safe if the function S_n gives the same weight to configurations that differ by a soft gluon, or that are identical up to the collinear splitting of a hard parton
  - Indeed, all the cuts through a potentially dangerous loop correction in the forward amplitude have the same weight, and the KLN cancellation works in the same manner as in the completely inclusive case

Specific hadrons in the final state

- When considering a specific hadron in the final state, one needs a fragmentation function $D_{H/i}(z, \mu^2)$, which represent the probability to obtain the hadron H from the parton i with a momentum fraction z
- Again, such a probabilistic description is possible thanks to the incoherence of the hadronization process with respect to the hard scattering :
  - The process of hadronization occurs over timescales which are large compared to that of hard processes
  - Moreover, the hadronization of a particular parton does not depend on the other hard partons produced in the event
- The resummation of leading logarithms leads to a scale dependence of the fragmentation functions, which obey a DGLAP equation

Lecture III : QCD on the light-cone

- Light-cone coordinates - Infinite Momentum Frame
- Poincaré algebra on the light-cone - Galilean sub-algebra
- Canonical quantization on the light-cone
- Scattering by an external potential
- Light-cone QCD

Lecture IV : Saturation and CGC

- BFKL equation
- Saturation of parton distributions
- Balitsky-Kovchegov equation
- Color Glass Condensate - JIMWLK
- Analogies with reaction-diffusion processes
- Pomeron loops

Lecture V : Calculating observables

- Field theory coupled to time-dependent sources
- Generating function for the probabilities
- Average particle multiplicity
- Numerical methods for nucleus-nucleus collisions
  - Gluon production
  - Quark production