C. BFKL Evolution & The Small–x Problem

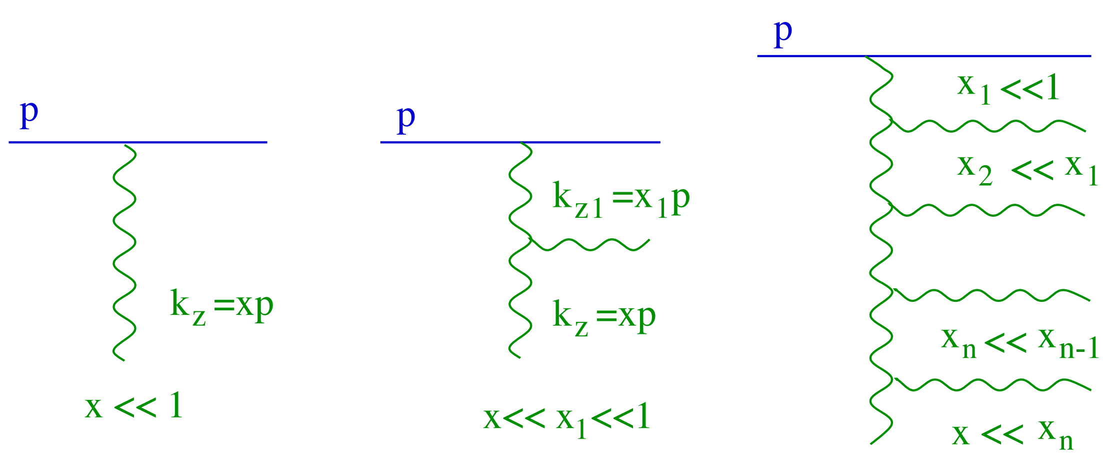
- Resum all terms $\left(\alpha_s \ln \frac{1}{x}\right)^n$, $n \geq 1$, in $xG(x, Q^2)$, even if non–accompanied by powers of $\ln Q^2$.
- Local in x, but non–local in $k_\perp$
- Exponential increase with $Y \equiv \ln(1/x)$ ("rapidity")

$$Y \equiv \ln \frac{1}{x} \sim \ln \frac{s}{Q^2} \qquad \mathrm{d}Y = \frac{\mathrm{d}x}{x} \simeq \mathrm{d}k_z \mathrm{d}b_z$$

$$xG(x, Q^2) \equiv \frac{\mathrm{d}N}{\mathrm{d}Y}(x, Q^2) = \int^{Q^2} \mathrm{d}^2 k_\perp \frac{\mathrm{d}N}{\mathrm{d}Y \,\mathrm{d}^2 k_\perp}$$

= # of gluons per unit rapidity and localized within a transverse size $\Delta x_\perp \sim 1/Q$.

$N(Y)$: # of gluons produced after a rapidity evolution Y



$\mathcal{A}_0$ $\qquad \sim \alpha_s \ln \frac{1}{x}$ $\qquad \sim \frac{1}{n!}\left(\alpha_s \ln \frac{1}{x}\right)^n$

$$N(Y) \sim \mathcal{A}_0 \sum_{n \geq 0} \frac{n+1}{n!} \left(\alpha_s Y\right)^n \propto \mathrm{e}^{\omega \alpha_s Y} = \frac{1}{x^{\omega \alpha_s}}$$

A more physical argument

- Gluons in the BFKL cascade are coherent in time.

  The lifetimes of the virtual gluons are strongly ordered along the cascade: $\Delta t_i \sim \frac{2\,x_i\,P}{k_{i\perp}^2}$

  $\Delta t_1 \gg \Delta t_1 \gg \ldots \Delta t_n \gg \Delta t$

- All the gluons with $x' \gg x$ are frozen over the natural time scale Δt for dynamics at x.

- The last gluon is emitted coherently off the global color charge of the previously emitted $N(Y)$ gluons

- Random distribution of 'fast' $(x' \gg x)$ gluons

  $\longleftrightarrow$ Random color charge $Q^a = \sum_i Q_i^a$

  $$\langle Q^a \rangle = 0, \qquad \langle Q^2 \rangle_Y \equiv \Big\langle \Big(\sum_{i=1}^{N} Q_i^a \Big)^2 \Big\rangle_Y \sim g^2 N_c\, N(Y)$$

  Indeed: for one gluon $(gT^a)(gT^a) = g^2 N_c$

  The color charges of the $N(Y)$ gluons sum up incoherently.

- Assumption: After being emitted, gluons do not interact with each other.

- Probability for emitting a new gluon when $Y \to Y + \mathrm{d}Y$:

  $$\mathrm{d}P(Y) \propto g^2 N_c\, N(Y)\mathrm{d}Y \;\equiv\; \omega\alpha_s N(Y)\mathrm{d}Y$$

- The average # of gluons at rapidity $Y + \mathrm{d}Y$:

$$N(Y+\mathrm{d}Y) \,=\, \big[1 + N(Y)\big]\mathrm{d}P(Y) \,+\, N(Y)\big[1 - \mathrm{d}P(Y)\big]$$

$$\Longrightarrow\; \frac{\mathrm{d}N}{\mathrm{d}Y} \,=\, \frac{\mathrm{d}P}{\mathrm{d}Y} \,=\, \omega\alpha_s N(Y) \Longrightarrow\; N(Y) \propto \mathrm{e}^{\omega\alpha_s Y}$$

- Unstable growth of the gluon distribution !

  The radiated gluons act as <u>sources</u> for the emission of new gluons.

- BFKL equation provides the value of ω :

$$\omega \,=\, 4\ln 2(N_c/\pi) \,\approx\, 2.65$$

together with the $k_\perp$–spectrum of the emitted gluons.

$\Longrightarrow$ The first problem of BFKL evolution at small–x :

Too rapid rise of total cross–sections with $1/x$ (or s)

- 1) Unacceptable phenomenology
- 2) Conceptual difficulties

1) Phenomenological difficulties of BFKL

- How to compute F_2 when $\alpha_s \ln \frac{1}{x} > \ln Q^2$?

  Collinear factorization does not apply beyond DLA.

  New factorization schemes will be introduced later:

  i) $k_\perp$–factorization ; ii) dipole picture

- In any case, at small–x, F_2 is driven by the gluon distribution, so BFKL generically implies :

  $F_2(x.Q^2) \propto 1/x^{\omega\alpha_s}$ with $\omega\alpha_s \simeq 0.5$ for $\alpha_s = 0.2$

- The data do rise indeed, but much slower !

  Parametrization : $F_2(x, Q^2) \sim x^{-\lambda_{\rm eff}(Q^2)}$, $x < 0.01$

  HERA data: $\lambda_{\rm eff}(Q^2)$ falls from 0.35 to 0.1 when

  Q^2 falls from 200 GeV^2 to 1 GeV^2.

- Similar conclusions from other high energy experiments ($\gamma^*\gamma^*$ scattering at L3, OPAL):

  The "BFKL intercept" $\alpha_P - 1 \equiv \omega\alpha_s$ is far too large!

- Next–to–leading order (NLO) corrections to BFKL could reduce α_P significantly.

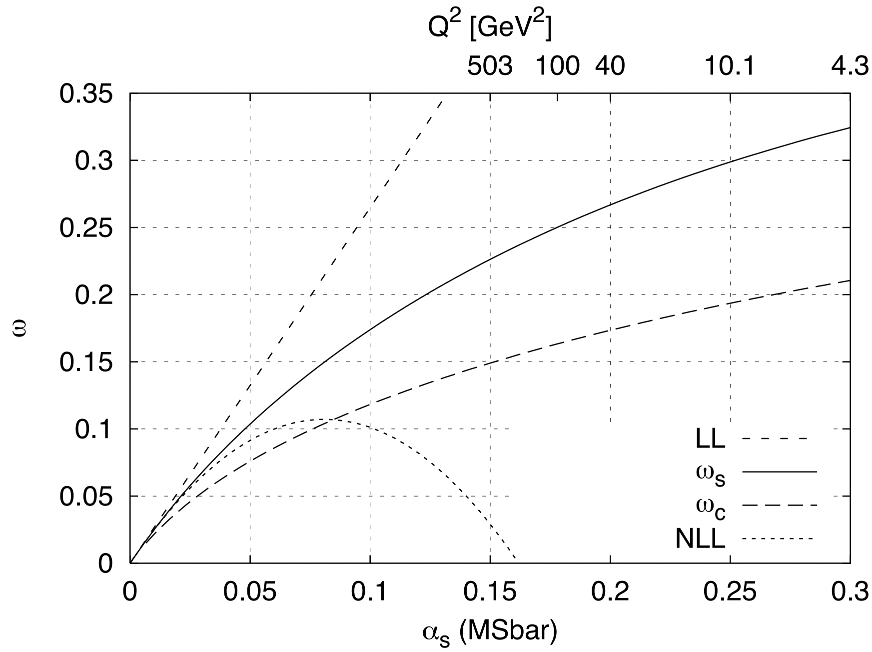
- NLO BFKL become available recently:

  (Fadin, Lipatov (98); Ciafaloni, Camici (98))

  and turned out to be larger then the leading–order !

  $\omega_{\rm NLO} = \omega_{\rm LO}(1 - 6.47\bar{\alpha}_s) = -0.16$ for $\alpha_s = 0.2$

- Resummations of higher–order effects have been proposed to cure this lack–of–convergence problem : Salam, 98; Ciafaloni, Colferai, 99; Brodsky, Fadin, Kim, Lipatov, Pivovarov, 99; Schmidt, 99; Forshaw, Ross, Sabio–Vera, 99
- E.g.: The collinear resummation by Salam, Ciafaloni, and Colferai $\Longrightarrow$ sensible results for $\omega_{\rm NLO}$:



From G. Salam, hep-ph/9910492

- Applications of NLO BFKL to phenomenology. Currently, a very active, and promising, field:
  — $\gamma^*\gamma^*$ scattering: encouraging results
  — DIS: recent progress, but some problems remain to be solved [Bartels, Colferai, Gieseke, Kyrieleis, Qiao]

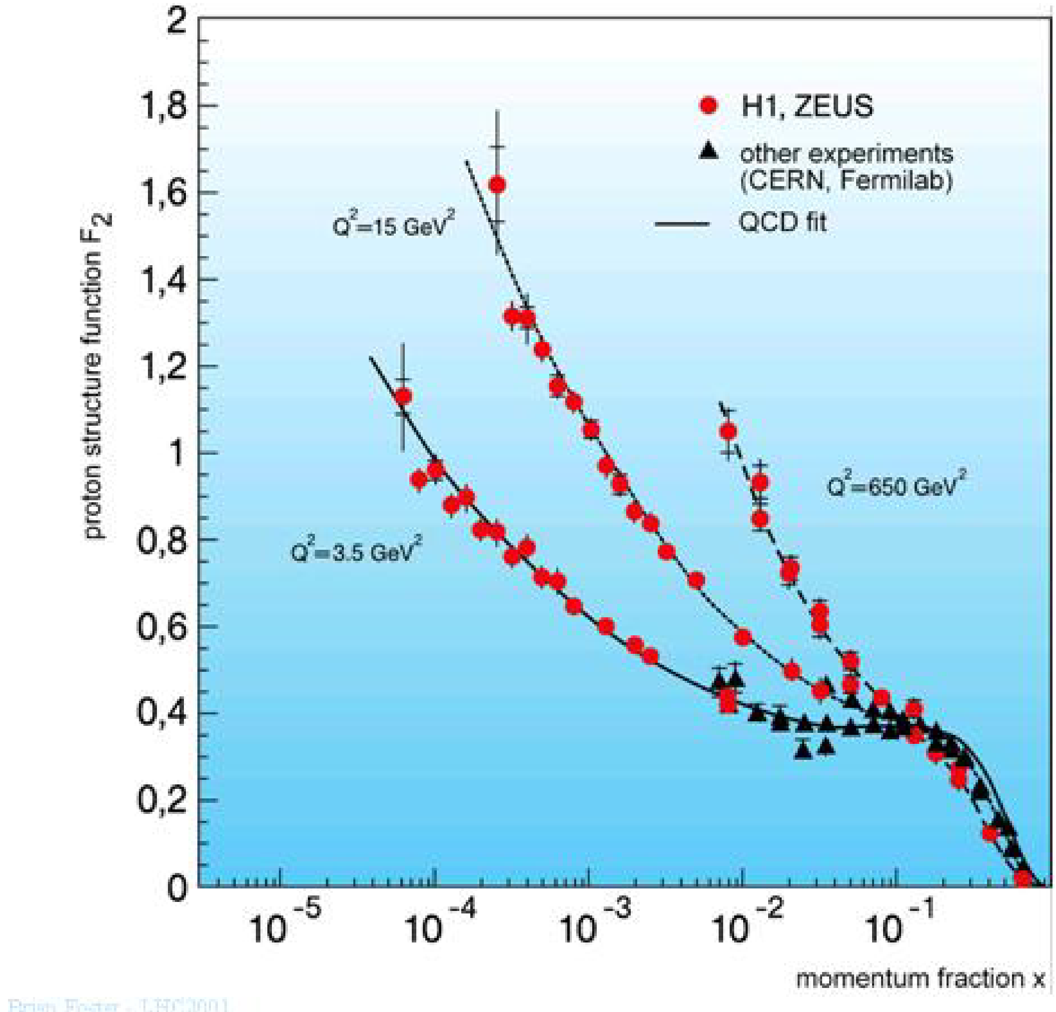


Figure 1: H1 and ZEUS data on the F_2 structure function shown in three bins of Q^2 as a function of x. The steep rise of the structure function at low x is clearly apparent.

From the review paper "Lectures on HERA physics", by B. Foster, EPJdirect A1, 1–11 (2003); hep-ex/0206011.

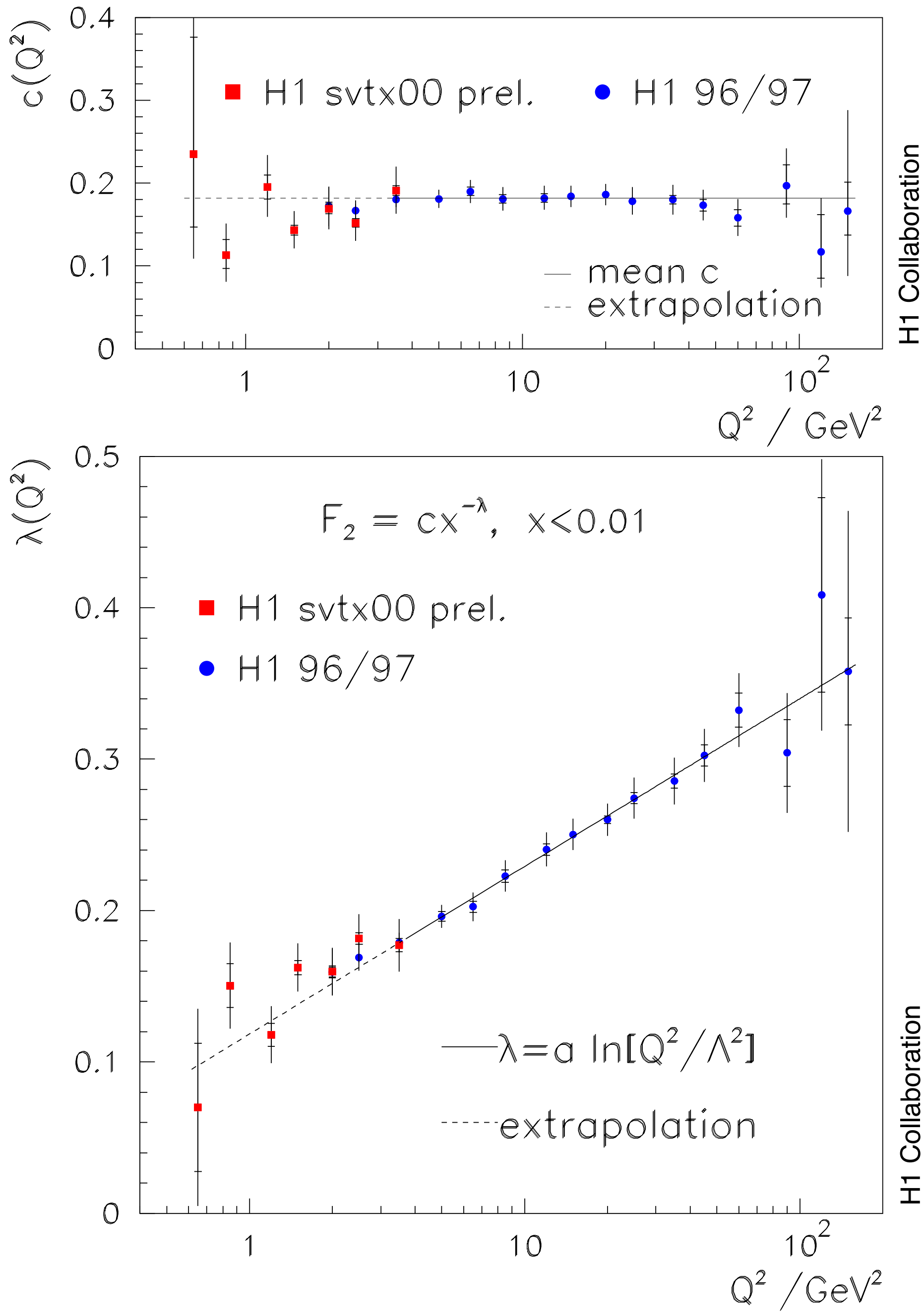


Figure 2: The coefficients $c(Q^2)$ and $\lambda(Q^2)$ from fits of the form $F_2(x, Q^2) = c(Q^2)x^{-\lambda(Q^2)}$ for $x < 0.01$

H1prelim-02-041, T. Lastovicka, talk presented at DIS 2002, April 2002 & H1 contribution to EPS2003, Aachen, July 2003

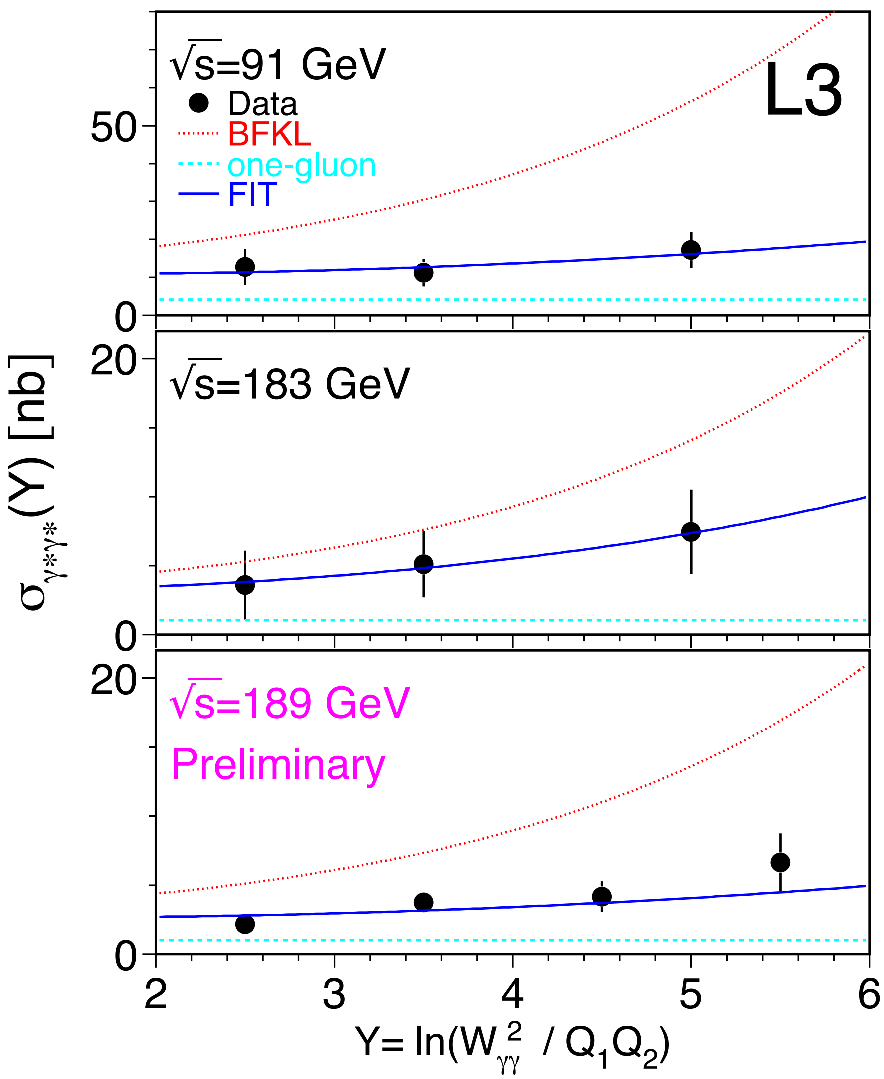


Figure 3: L3 results for the two–photon cross–section $\sigma_{\gamma^*\gamma^*}$ showing a strong deviation from LO BFKL. The continuous line is a BFKL–like fit with $\alpha_P - 1$ left as a free parameter. One finds: $\alpha_P - 1 \approx 0.28$ for $\sqrt{s} = 91$ GeV and $\alpha_P - 1 \approx 0.40$ for $\sqrt{s} = 183$ GeV.

From the review paper by Donnachie, J. Phys. G26: 689-695, 2000 hep-ph/0001035

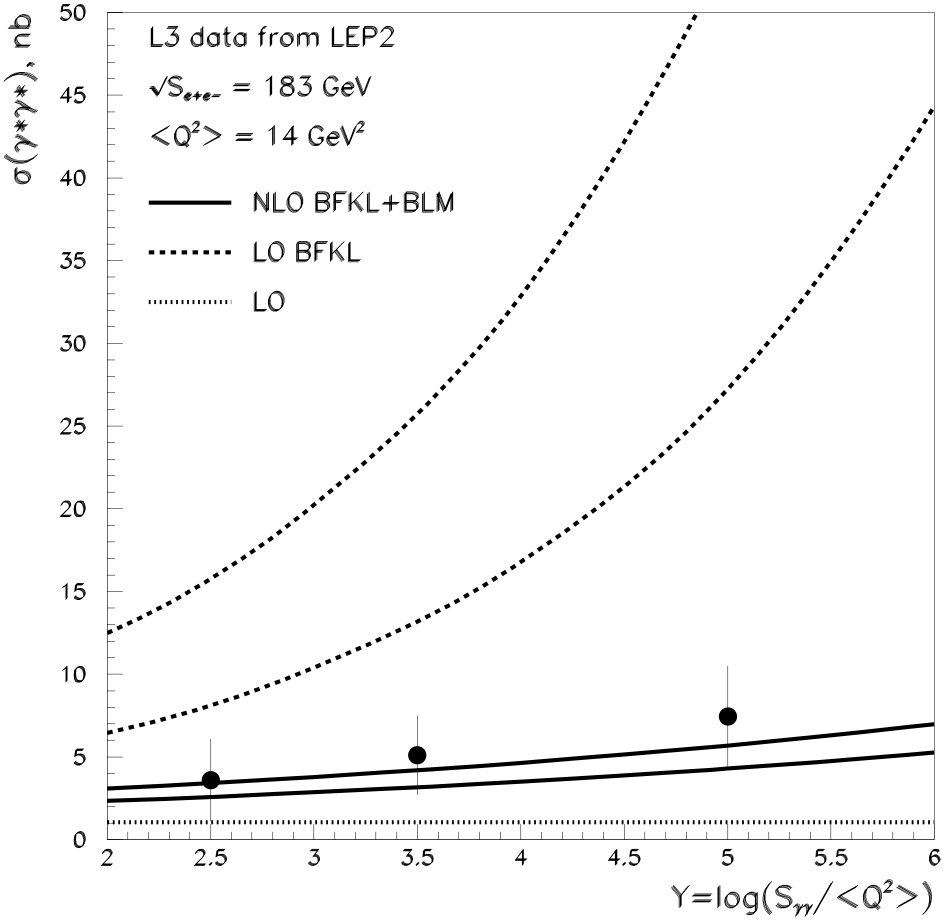


Figure 4: Virtual gamma-gamma total cross section by the NLO BFKL Pomeron vs L3 Collaboration data at energy 183 GeV of e^+e^- collisions. Solid curves: NLO BFKL. Dashed: LO BFKL. Dotted: LO contribution. Two different curves are for two different choices of the Regge scale: $s_0 = Q^2/2$, $s_0 = 2Q^2$

From Kim, Lipatov, Pivovarov, hep-ph/9911228, presented to the VIIIth Blois Workshop, 1999

2) Unitarity violation by BFKL evolution

$$\sigma_{\gamma^* p}(x, Q^2) \,=\, \frac{4\pi^2\alpha_{\rm em}}{Q^2}\, F_2(x, Q^2) \propto \frac{1}{x^{\omega\alpha_s}} \sim s^{\omega\alpha_s}$$

- But hadronic crosss-sections cannot grow like a power of s in the high–energy limit $s \to \infty$!
- (A) Froissart bound: $\sigma_{\rm tot}(s) \,\le\, \sigma_0 \,\ln^2 s$ as $s \to \infty$.
  Consequence of very general principles (unitarity, analitycity, crossing) + short–rangeness (mass gap)
  In QCD one expects: $\sigma_0 \propto 1/m_\pi^2$.
- DIS: γ^* is a virtual state, not a hadron !
  Still: γ^* is a superposition of hadronic states

$$\Longrightarrow \qquad \sigma_{\gamma^* p}(x, Q^2) \,\le\, \sigma_0 \,\ln^3 \frac{1}{x} \qquad \text{as} \qquad x \to 0$$

- Clearly, BFKL (LO or NLO) violates this condition.
- (B) Unitarity bound on the S–matrix : $SS^\dagger \,=\, 1$
  $\Longrightarrow |S(x, b)| \,\le\, 1$ for any impact parameter b.
  This condition too is violated by BFKL (see below).
- Is this a problem of the BFKL evolution of $xG(x, Q^2)$, or of the calculation of F_2, or both?
  One expects multiple scattering to restore unitarity in the S–matrix.
  One may imagine that F_2 unitarizes, while $xG(x, Q^2)$ keeps growing as a power of $1/x$.

The BFKL equation

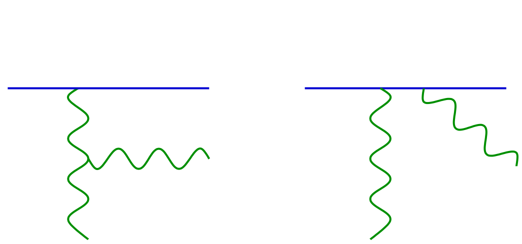
- The unintegrated gluon distribution (with $\boldsymbol{k} \equiv k_\perp$) :

$$xG(x,Q^2) \equiv \int^{Q^2} \frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2}\, f(x,\boldsymbol{k}^2), \quad f(x,\boldsymbol{k}^2) = \frac{\partial\, xG(x,\boldsymbol{k}^2)}{\partial \ln \boldsymbol{k}^2}$$

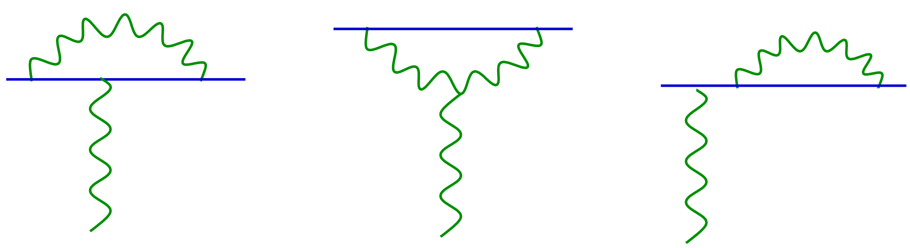
- BFKL equation ($Y = \ln(1/x)$ and $\bar{\alpha}_s = \alpha_s N_c/\pi$)

$$\frac{\partial f(Y,\boldsymbol{k}^2)}{\partial Y} = \bar{\alpha}_s \int \frac{\mathrm{d}^2\boldsymbol{p}}{\pi} \frac{\boldsymbol{k}^2}{\boldsymbol{p}^2(\boldsymbol{k}-\boldsymbol{p})^2} \Big\{ \underbrace{f(Y,\boldsymbol{p}^2)}_{\text{real}} - \underbrace{\frac{1}{2} f(Y,\boldsymbol{k}^2)}_{\text{virtual}} \Big\}$$

- "Real" : real gluon emission in one–step evolution



- "Virtual" : vertex and self-energy corrections



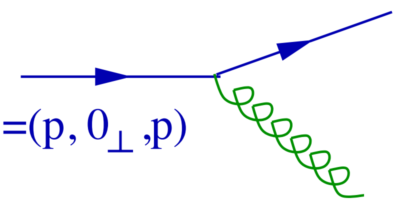
- The virtual term compensates the "infrared' divergence of the real term at $\boldsymbol{k} = \boldsymbol{p}$.

- DLA limit comes out as expected: for $\boldsymbol{k}^2 \gg \boldsymbol{p}^2$

$$\frac{\partial f(Y,\boldsymbol{k}^2)}{\partial Y} \approx \bar{\alpha}_s \int^{\boldsymbol{k}^2} \frac{\mathrm{d}\boldsymbol{p}^2}{\boldsymbol{p}^2}\, f(Y,\boldsymbol{p}^2)$$

The Gluon distribution of one quark

- Recall: gluon radiation by a fast quark ('bremsstrahlung')



$$\mathrm{d}\mathcal{P} \,=\, \frac{\alpha_s C_F}{2\pi}\,\frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2}\,\frac{1+(1-x)^2}{x}\,\mathrm{d}x$$

where $C_F \equiv t^a t^a = (N_c^2-1)/N_c = 4/3$

- One can identify this with the spectrum of the emitted gluons :

$$\frac{\mathrm{d}N}{\mathrm{d}x\mathrm{d}\boldsymbol{k}^2} \,\equiv\, \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x\mathrm{d}\boldsymbol{k}^2} \,\simeq\, \frac{\alpha_s C_F}{\pi}\,\frac{1}{\boldsymbol{k}^2}\,\frac{1}{x} \quad \text{for} \quad x \ll 1$$

- Hence, the integrated gluon distribution generated by a single quark :

$$f_q(x,\boldsymbol{k}^2) \equiv \frac{\partial\, xG(x,\boldsymbol{k}^2)}{\partial \ln \boldsymbol{k}^2} = \boldsymbol{k}^2\, x\, \frac{\mathrm{d}N}{\mathrm{d}x\mathrm{d}\boldsymbol{k}^2} = \frac{\alpha_s C_F}{\pi}$$

- The (integrated) gluon distribution of one quark:

$$xG_q(x,Q^2) \,=\, \int^{Q^2} \frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2}\,\frac{\alpha_s C_F}{\pi} \,=\, \frac{\alpha_s C_F}{\pi}\,\ln\frac{Q^2}{\Lambda^2}$$

Λ = infrared cutoff

- The simplest "proton" : $xG_p(x,Q^2) \,\simeq\, 3\,xG_q(x,Q^2)$

- The solution to BFKL equation for $\alpha_s Y \gg \ln \boldsymbol{k}^2$

$$f(Y,\boldsymbol{k}^2) \propto \left(\frac{\boldsymbol{k}^2}{\boldsymbol{k}_0^2}\right)^{1/2} \mathrm{e}^{\omega \bar{\alpha}_s Y} \exp\left\{-\frac{\ln^2\left(\boldsymbol{k}^2/\boldsymbol{k}_0^2\right)}{2\beta\bar{\alpha}_s Y}\right\}$$

where: $\omega = 4\ln 2 \approx 2.77$, $\beta \approx 33.67$.

- Exponential rise with Y.

- BFKL "anomalous dimension 1/2": $f \propto \sqrt{\boldsymbol{k}^2}$

  In LO perturbation th., f is independent of $\boldsymbol{k}$.

  DGLAP evolution: f is a function of $\ln \boldsymbol{k}^2$.

BFKL: strong violation of Bjorken scaling: $F_2 \propto \sqrt{Q^2}$

- Diffusive behaviour in the variable $\ln \boldsymbol{k}^2$ with diffusion "time" Y.

  The evolution broadens the $\ln \boldsymbol{k}^2$–distribution of f.

$\Longrightarrow$ New problem of BFKL evolution at small–x :

"Infrared diffusion" towards non–perturbative momenta

Even if the external momentum $\boldsymbol{k}$ is hard ($\gg \Lambda^2_{\mathrm{QCD}}$), the non-locality of the evolution makes it that, with increasing Y, $f(Y,\boldsymbol{k}^2)$ receives an increasingly large contribution from non–perturbative $\boldsymbol{p}$.

- Problems with the introduction of the running coupling $\alpha_s(\boldsymbol{k}^2)$

- In practice: use an infrared cutoff $\boldsymbol{k}_c^2$

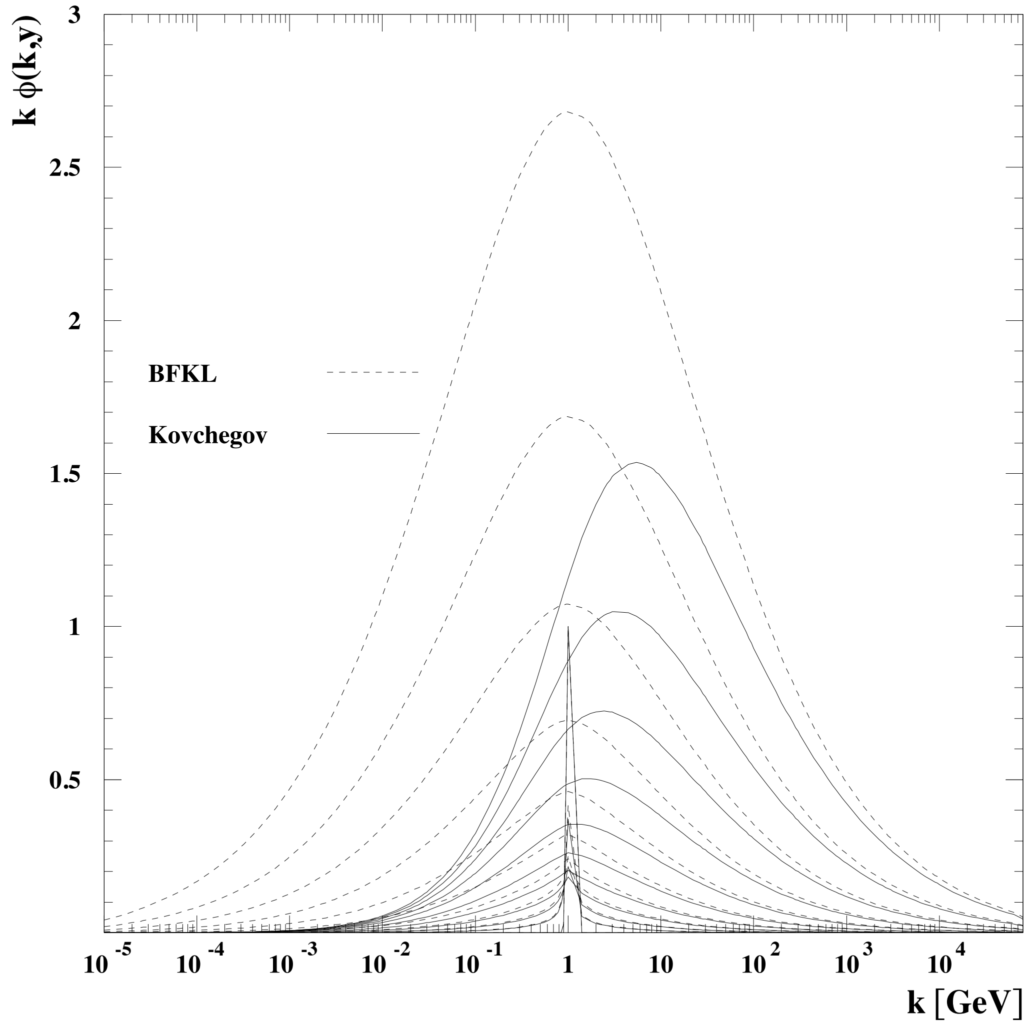


Figure 5: The functions $k\phi(k,Y)$ constructed from solutions to the BFKL and the Balitsky-Kovchegov equations for different values of the evolution parameter $Y = \ln(1/x)$ ranging from 1 to 10. The coupling constant $\alpha_s = 0.2$.

$$k\phi(k,Y) \;\longleftrightarrow\; f(Y,k)/k; \qquad k\phi(k,Y_0=0) \;=\; \delta\big(\ln k^2/k_0^2\big)$$

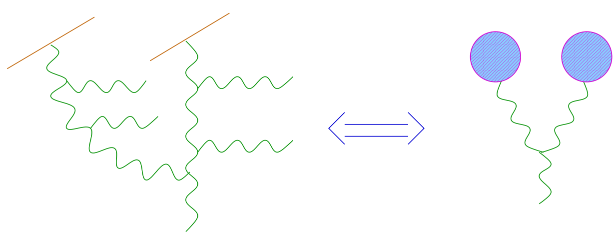
From K. Golec-Biernat, L. Motyka, A. M. Stasto, Phys Rev D65 (2002) 074037; hep-ph/0110325

III. Unitarity vs. Saturation in DIS

A. The idea of saturation

Gribov, Levin and Ryskin, 83; Mueller and Qiu, 86

- So far, the emitted gluons were assumed not to interact with each other
- Gluons within the same BFKL cascade are well separated in rapidity, so they cannot interact
- Gluons with similar rapidities but from different cascades interact with a cross–section $\sim \alpha_s$
- At small–x, the gluon density is large, which enhances gluon recombination



When RECOMBINATION = RADIATION $\Longrightarrow$ SATURATION

- What is the recombination probability ?
- In order to interact, the small–x gluons must overlap in transverse projection.

  Indeed, since $\lambda_z \sim \frac{1}{xP} \gg \frac{1}{P}$, their longitudinal overlap is then automatic.

  $xG(x, Q^2)/\pi R^2$ = # of gluons (x, Q^2) per unit area.

- A typical gg cross-section: $\sigma_{gg}(Q^2) \sim \alpha_s/Q^2$
- The probability that a given gluon interacts with all other gluons with the same (x, Q^2) :

$$\Gamma(x, Q^2) \simeq \frac{\alpha_s(Q^2)}{Q^2}\,\frac{xG(x,Q^2)}{\pi R^2}$$

i) $O(\alpha_s)$, but amplified by $xG(x,Q^2) \sim x^{-\omega}$.

ii) Suppressed by $1/Q^2$: "higher twist".

- $\Gamma(x, Q^2) \sim 1$ for $Q^2 \sim Q_s^2(x)$ such that :

$$Q_s^2(x) \simeq \alpha_s\,\frac{xG(x,Q_s^2(x))}{\pi R^2} \sim x^{-\lambda}$$

$Q_s(x) \gg \Lambda_{\rm QCD}$ provided x is small enough !

$\Longrightarrow$ A hard intrinsic scale which may suppress infrared diffusion and ensure weak coupling :

$$\boxed{\alpha_s\big(Q_s^2(x)\big) \ll 1 \quad \text{for} \quad x \ll 1}$$

- $Q_s(x)$ = "saturation momentum"

— For $k_\perp \le Q_s(x)$: strong non–linear effects which should saturate the gluon occupation number:

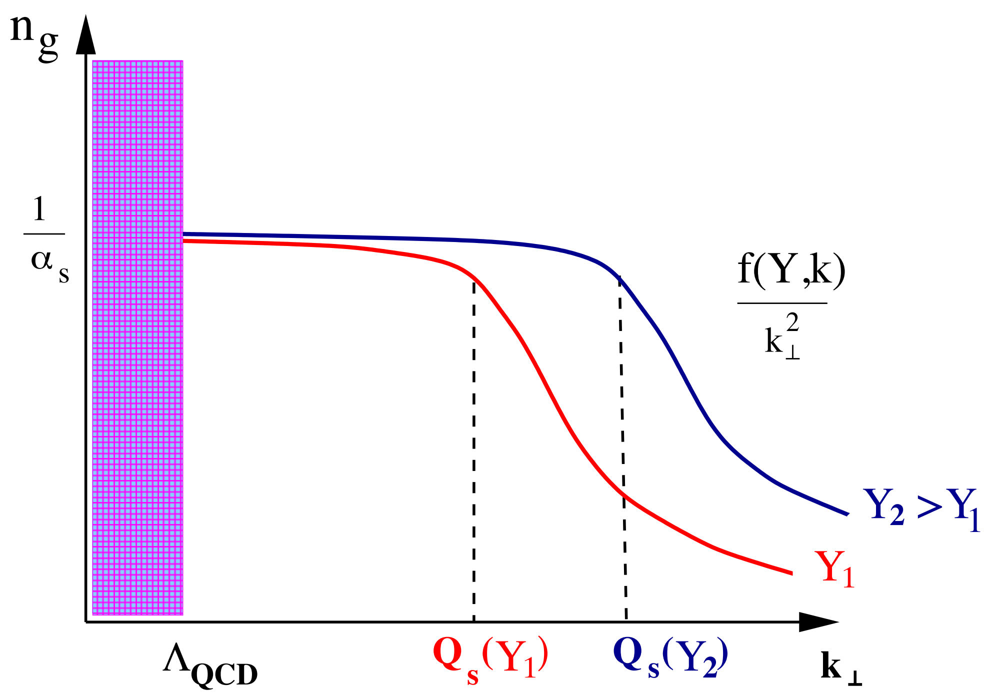
$$n_g \equiv \frac{(2\pi)^3}{2\cdot(N_c^2-1)}\,\frac{{\rm d}N}{{\rm d}Y\,{\rm d}^2\boldsymbol{k}\,{\rm d}^2\boldsymbol{b}} \sim \frac{1}{\alpha_s}$$

— For $k_\perp \gg Q_s(x)$: usual linear evolution.

$n_g \ll 1$, but rapidly increasing with $1/x$ and $1/k_\perp$

With decreasing x, gluons are produced mostly at large momenta $k_\perp > Q_s(x)$, and thus cannot be "seen" by a probe (γ^*) with $Q^2 < Q_s(x)$

$\Longrightarrow$ unitarization of DIS!

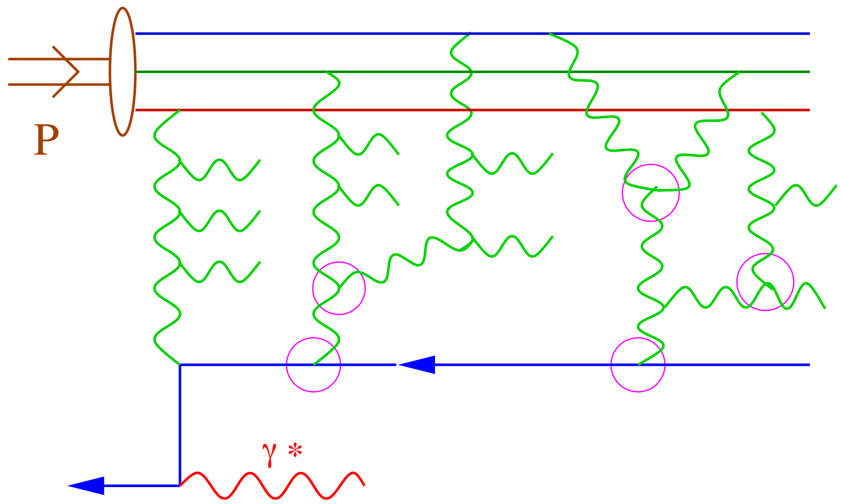


- Gluon saturation is a natural solution to the small–x problems of the linear evolution equations.
- How to compute in this non–linear regime ?

  Large occupation numbers $\longleftrightarrow$ Strong ($A \sim 1/g$) classical fields. (cf. Introduction)

Classical effective theory for the small–x gluons

A cartoon of DIS in the presence of non–linear effects



- Non–linearities manifest themselves is:

  — the gluon distribution, and its evolution with $1/x$

  — the production of the last quark

- Strategy :

  1) 'Factorize out' the quark production and its subsequent interaction with γ^* (specific to DIS)

  "Dipole Picture" + "Eikonal Approximation"

  A. Mueller; Nikolaev, Zakharov; Kovchegov

  Buchmüller, Hebecker; Balitsky

  2) Construct an <u>effective theory</u> for gluon correlations in the hadron wavefunction at small–x

  "Color Glass Condensate"

  McLerran, Venugopalan: "MV model" (classical model for a large nucleus $A \gg 1$)

  E.I., Jalilian-Marian, Kovner, Leonidov, McLerran, Weigert: QCD at small x

B. Light–cone kinematics & quantization

- A fast moving particle in the positive z direction: $p_z \gg m$, $v \simeq c$: $z = t$, $x, y = \text{const.}$
- Light–cone coordinates:

$$x^+ \equiv \frac{1}{\sqrt{2}}(t+z), \quad x^- \equiv \frac{1}{\sqrt{2}}(t-z), \quad x_\perp = (x, y)$$

- "Particle at rest at $x^- = 0$ with 'time' x^+"
- For a particle moving in the negative z direction ($z = -t$), the roles of x^+ and x^- get interchanged.
- 4–vector: $x^\mu = (x^+, x^-, x_\perp)$

  Measure: $\mathrm{d}^4x = \mathrm{d}x^+\mathrm{d}x^-\mathrm{d}^2x_\perp$

  Metric: $k \cdot x = k^-x^+ + k^+x^- - k_\perp \cdot x_\perp$

k^-: "LC energy"; k^+: "LC longitudinal momentum"

$$\text{on}-\text{shell excitation}: \quad k^2 = m^2 \quad \Longrightarrow \quad k^- = \frac{m^2 + k_\perp^2}{2k^+}$$

Proton's IMF: $P_z \equiv P \gg M \Longrightarrow P^\mu \simeq \delta^{\mu +} P^+$

$P^+ \simeq \sqrt{2}P$, $P^- = M^2/2P^+ \ll P^+$

- Boost–invariant longitudinal momentum fraction:

$$x \equiv \frac{k^+}{P^+} \quad \left(\longrightarrow \frac{k_z}{P_z} \quad \text{in IMF}\right)$$

and rapidity: $Y = \ln(1/x) = \ln(P^+/k^+)$

Gluon Distribution

$$xG(x,Q^2)\,\equiv\,\int^{Q^2}\mathrm{d}^2\boldsymbol{k}\,\frac{\mathrm{d}N}{\mathrm{d}Y\mathrm{d}^2\boldsymbol{k}}$$

- LC quantization of the SU(3) gauge fields

  LC gauge: $A_a^+=0$ ($\longleftrightarrow$ temporal gauge $A_a^0=0$)

- Fock–space gluon number density:

$$\frac{\mathrm{d}N}{\mathrm{d}^3k}\equiv\frac{\mathrm{d}N}{\mathrm{d}k^+\mathrm{d}^2\boldsymbol{k}}=\sum_{\lambda,c}\big\langle a^\dagger_{\lambda,c}(\vec{k})a_{\lambda,c}(\vec{k})\big\rangle\qquad[\vec{k}\equiv(k^+,\boldsymbol{k})]$$

- One finds (with $F_a^{+i}(x)=\partial^+A_a^i(x)$ in LC gauge) :

$$\frac{\mathrm{d}N}{\mathrm{d}Y\mathrm{d}^2\boldsymbol{k}}\,=\,\frac{1}{4\pi^3}\big\langle F_a^{+i}(x^+,\vec{k})F_a^{+i}(x^+,-\vec{k})\big\rangle$$

  where $\langle\cdots\rangle$: average over hadron wavefunction.

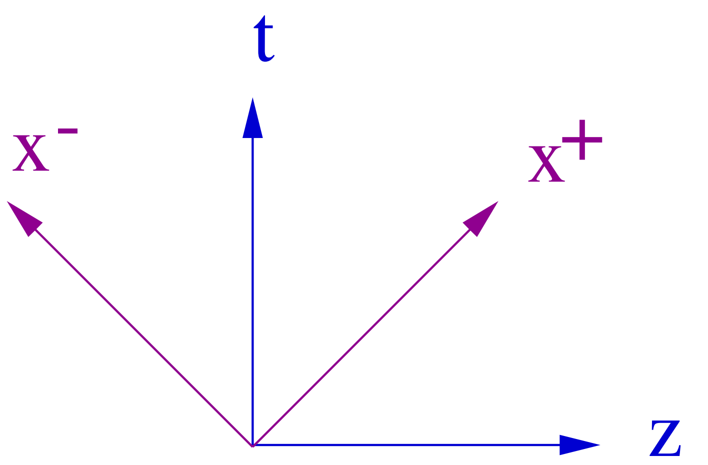
- This is a gauge–invariant quantity, but coincides with the Fock–space gluon number only in LC gauge.

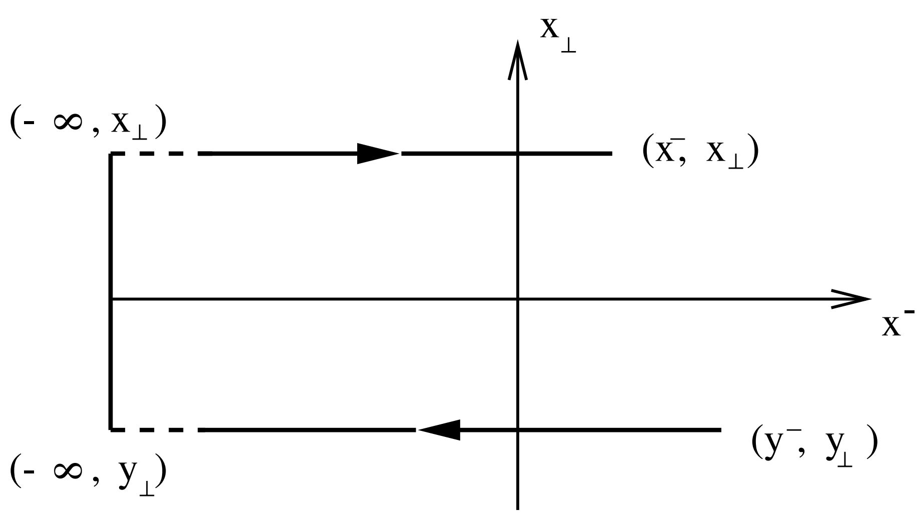
- In some arbitrary gauge, the r.h.s. is replaced by $\langle\mathcal{O}_\gamma\rangle$, with the gauge–invariant operator:

$$\mathcal{O}_\gamma(\vec{x},\vec{y})\equiv\ \mathrm{Tr}\,\Big\{F^{i+}(\vec{x})\,U_\gamma(\vec{x},\vec{y})\,F^{i+}(\vec{y})\,U_\gamma(\vec{y},\vec{x})\Big\}$$

  where $\vec{x}=(x^-,\boldsymbol{x})$ and the Wilson line:

$$U_\gamma(\vec{x},\vec{y})\,=\,\mathrm{P}\exp\left\{ig\int_\gamma\mathrm{d}\vec{z}\cdot\vec{A}_a(\vec{z})T^a\right\}$$





The path γ used for the evaluation of the gauge-invariant operator $\mathcal{O}_\gamma$.

- Gauge transformations: $h(x) \in \mathrm{SU}(3)$, $hh^\dagger = 1$

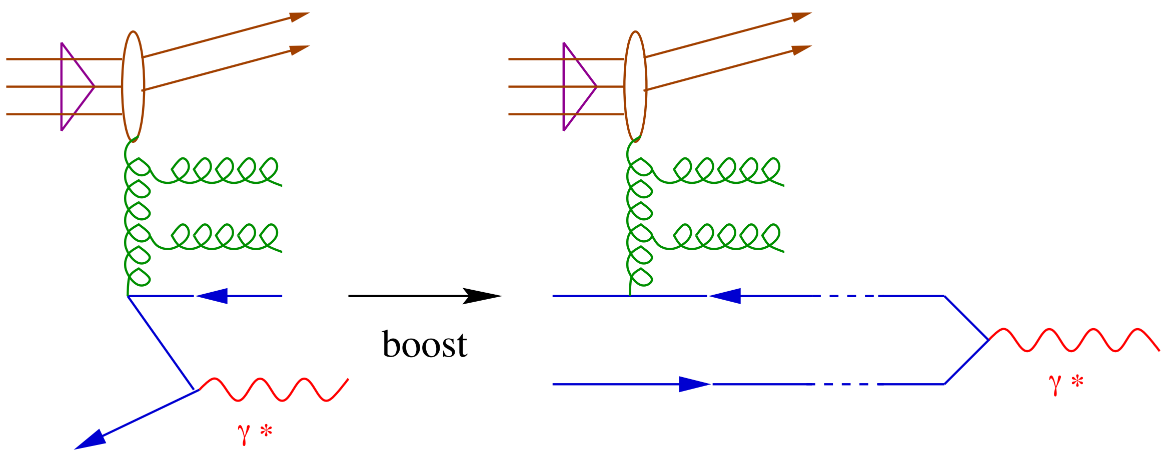
$$A^\mu(x) \equiv A^\mu_a(x)T^a \longrightarrow h(x)A^\mu(x)h^\dagger(x) + \frac{i}{g}\, h(x)\partial^\mu h^\dagger(x)$$

$$U_\gamma(x,y) \longrightarrow h(x)\, U_\gamma(x,y)\, h^\dagger(y)$$

C. DIS in the Dipole Frame

- How to compute F_2 in the presence of "higher–twist" effects ?
- The struck quark: the last emitted parton in an otherwise purely gluonic cascade
- It is convenient/possible to disentangle this last quark from the gluon evolution via a Lorentz boost :

  Give γ^* enough energy to fluctuate into a $q\bar{q}$ pair long time before the collision:



$$|\gamma^*\rangle \,=\, c_0|\gamma\rangle_0 \,+\, c_1|q\bar{q}\rangle_0 + \,\cdots \qquad (c_1 \sim \alpha_{\rm em})$$

$|q\bar{q}\rangle_0$: "Color Dipole" (color single state);

- To $O(\alpha_{\rm em})$, DIS is determined by the dipole scattering off the color fields in the hadron.

- Dipole frame :

  $P^\mu \simeq (P, 0_\perp, P) \quad \& \quad q^\mu = (\sqrt{q^2 - Q^2}, 0_\perp, -q)$

  with $q \gg Q$ but such that $\alpha_s \ln(q/Q) \ll 1$

  i) The $q\bar{q}$ fluctuation has a long lifetime:

  $\tau_{q\bar{q}} \sim \big(E_{q\bar{q}} - E_{\gamma^*}\big)^{-1} \gg \tau_{\rm DIS} \sim R/\gamma \sim 1/P$

  $\Longrightarrow$ the dipole transverse size $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ is frozen (unchanged by the dipole–hadron interaction).

  ii) No gluon (QCD evolution) in the $q\bar{q}$ wavefunction.

  Most of the total energy, and all the evolution, are still carried by the proton !
  $\alpha_s \ln(1/x) \simeq \alpha_s \ln(P/Q) > 1$

- Small–x DIS in the Dipole Frame :

  Dissociation of γ^* into a $q\bar{q}$ pair followed (long time after) by the interaction between a bare color dipole and a highly evolved hadron.

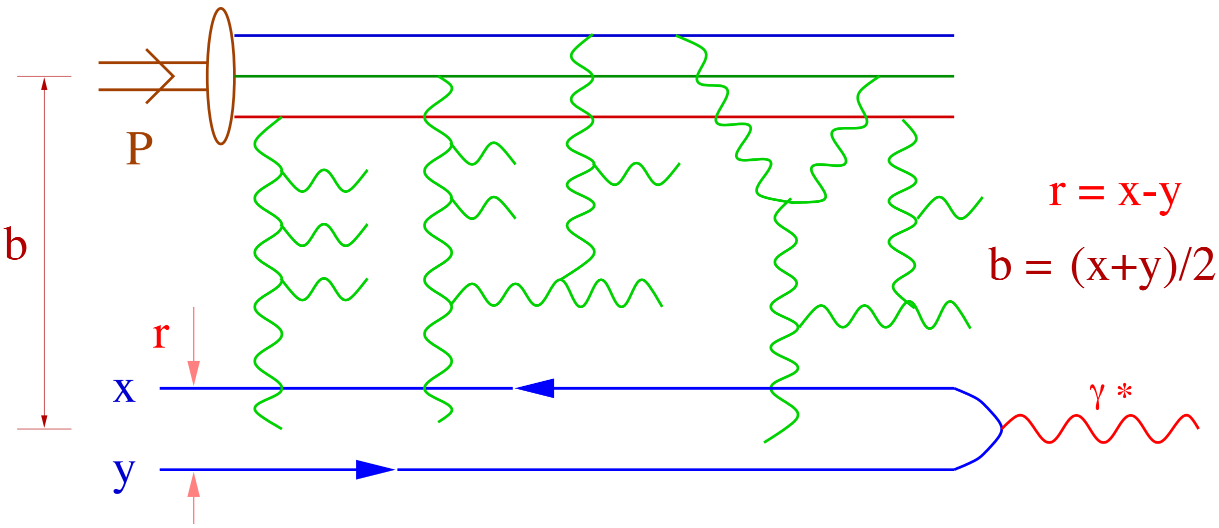
  $\sigma_{\gamma^* p}(x, Q^2) = \sigma_T + \sigma_L$

$$\sigma_{T,L}(x, Q^2) = \int_0^1 dz \int d^2\boldsymbol{r} \, \big|\Psi_{T,L}(z, \boldsymbol{r}; Q^2)\big|^2 \, \sigma_{\rm dipole}(x, \boldsymbol{r})$$

- $|\Psi|^2 \sim O(\alpha_{\rm em})$: LC wavefunction for the dissociation of γ^* into a $q\bar{q}$ pair of size $\boldsymbol{r}$ and a longitudinal fraction of the quark equal to z

- All the QCD dynamics is encoded in $\sigma_{\rm dipole}$.

- How to compute $\sigma_{\rm dipole}$?



- Eikonal approximation:

Relative motion is very fast ($s \sim qP \gg k_\perp^2$)

— straightline trajectories ($x_\perp$, $y_\perp$ = fixed)

— coupling to A_a^+ alone: $j_q^\mu A_\mu \propto A^+$ (since q and $\bar{q}$ propagate in the negative z direction: $j_q^\mu \propto \delta^{\mu -}$)

(Use a gauge <u>different</u> from LC gauge $A^+ = 0$!)

$$\sigma_{\rm dipole}(x, \boldsymbol{r}) \,=\, 2\int {\rm d}^2\boldsymbol{b}\left(1 - {\rm Re}\, S(x, \boldsymbol{r}, \boldsymbol{b})\right)$$

$S(x, \boldsymbol{r}, \boldsymbol{b})$: S–matrix at impact parameter $\boldsymbol{b}$.

$|S| \leq 1$ (unitarity bound, from $SS^\dagger = 1$)

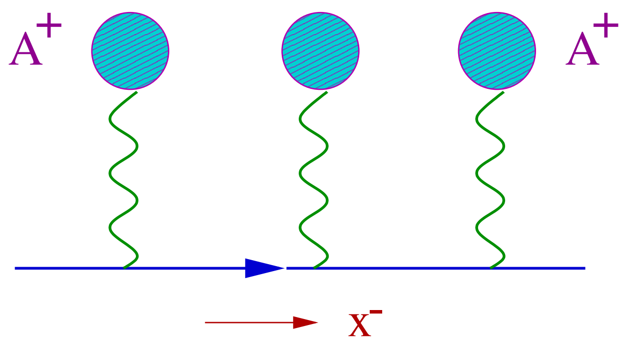
$S \approx 1$: weak scattering (transparency);

$S \ll 1$: complet absorbtion (blackness)

- The S–matrix in the eikonal approximation:

  i) Single quark in the background of the field A_a^+ :

$$S_q(\boldsymbol{x}) \,=\, \mathrm{P}\, \exp\left\{ ig \int \mathrm{d}x^- A_a^+(x^-, \boldsymbol{x}) t^a \right\} \,\equiv\, V^\dagger(\boldsymbol{x})$$



  ii) Color dipole in the fluctuating field of a hadron:

$$S_Y(\boldsymbol{x}, \boldsymbol{y}) \,=\, \frac{1}{N_c} \left\langle \mathrm{tr}\, V^\dagger(\boldsymbol{x})\, V(\boldsymbol{y}) \right\rangle_Y \,\equiv\, S_Y(\boldsymbol{r}, \boldsymbol{b})$$

- Schematically : $S \,=\, \langle \phi_f | \phi_i \rangle \,=\, \langle \phi_i | U(\infty, -\infty) | \phi_i \rangle$

  $U(\infty, -\infty) \,=\, \mathrm{T}\, \exp\left\{ i \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{L}_{\rm int}(t, \vec{x}) \right\}$

  Here: "time" $= x^-$, $\vec{x} = (x^+, \boldsymbol{b})$,

  $\mathcal{L}_{\rm int} = j_a^\mu A_\mu^a$ with $j_a^\mu = g t^a \delta^{\mu -} \delta(x^+) \delta(\boldsymbol{b} - \boldsymbol{x})$

- $(1/N_c)$ tr : average over color for the $q\bar{q}$ pair

- V, $V^\dagger \in \mathrm{SU}(3)$: "Wilson lines"

  Color precession of the quark (or antiquark) after crossing the field.

- V, $V^\dagger$: All orders in $gA^+ \Longrightarrow$ Multiple scattering

- Weak fields ($gA^+ \ll 1$) $\Longrightarrow$ Perturbative expansion

  Appropriate at not so high densities (not so small x).

- Lowest order in α_s : Single–scattering approximation

$$S_Y(\boldsymbol{x}, \boldsymbol{y}) \approx 1 - \frac{g^2}{4N_c} \Big\langle \Big(A_a^+(\boldsymbol{x}) - A_a^+(\boldsymbol{y}) \Big)^2 \Big\rangle_Y$$

$\langle A^+ A^+ \rangle \propto$ gluon distribution.

Exercice i) Show that to $O(\alpha_s)$:

$$\sigma_{\text{dipole}}(x, \boldsymbol{r}) \approx \frac{4\pi}{N_c} \alpha_s \int \frac{\mathrm{d}^2\boldsymbol{k}}{\boldsymbol{k}^4} f(x, \boldsymbol{k}^2) \left(1 - \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

ii) If $f(x, \boldsymbol{k}^2)$ is slowly varying in $\boldsymbol{k}^2$, show that:

$$\sigma_{\text{dipole}}(x, \boldsymbol{r}) \approx \frac{\pi^2}{N_c} \alpha_s \boldsymbol{r}^2 \, xG(x, 1/\boldsymbol{r}^2)$$

- The weak field/coupling expansion to a given order violates the unitarity bound.

- When $gA^+ \sim 1$, any number of collisions is important, and unitarity should be restored.

- When $gA^+ \sim 1$, non–linear effects in the gluon distribution become important too.

Unitarization of dipole scattering

& Saturation of the gluon distribution:

Two aspects of the same non–linear physics !

$$\sigma_{\text{dipole}}(x, r) \sim \pi R^2 \quad \text{when} \quad r \sim 1/Q_s(x)$$