ACADEMIC TRAINING LECTURES
DESY, November 17, 18 & 20, 2003

QCD AT HIGH ENERGIES

- What is the high energy limit of scattering in QCD ?
- What is a high energy hadron made of ?
- To which extent can this be described in perturbation theory ?
- A new regime of QCD: weak coupling & high density
- Color Glass Condensate
  — The form of hadronic matter which controls high–energy interactions
  — An effective theory for the small–x part of the wavefunction of an energetic hadron

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PLAN

- I. Introduction : High–Density QCD
  - A. High–Temperature QCD
  - B. High–Energy QCD
- II. Deep Inelastic Scattering at Small–x
  - A. Generalities (Kinematics, Structure functions, Parton picture)
  - B. Parton Evolution in QCD
  - C. BFKL Evolution and the Small–x Problem
- III. Unitarity vs. Saturation in DIS
  - A. The Idea of Saturation
  - B. Light–Cone Kinematics & Quantization
  - C. DIS in the Dipole Frame
- IV. The Color Glass Condensate
  - A. Effective Theory for the Small–x Gluons
  - B. The Classical Solution
  - C. The gluon distribution of the valence quarks (McLerran–Venugopalan model)
  - D. The Renormalization Group at Small x
  - E. General Consequences of the Renormalization Group Equation
  - F. Non–Linear Gluon Evolution: Saturation & Geometric Scaling

- V. DIS off the CGC

  A. Balitsky–Kovchegov Equation

  B. Unitarization & Geometric Scaling for
     Dipole Scattering

  C. The Golec–Biernat & Wüsthoff Saturation Model

  D. A CGC fit to the F_2 data at HERA

- VI. Perspective and Open Problems

  Total cross-section and Froissart bound

  Physics at RHIC

I. High–Density QCD

- A medium described by QCD

  A dense system of quarks and gluons which exists over time scales much larger than the time scale we need to probe it.

  — Quark–Gluon Plasma

  (Early Universe, Heavy Ion collisions)

  — Color Glass Condensate (High energy scattering)

  — Color Superconductors (Core of neutron stars)

- Weak coupling but many degrees of freedom

  $\Longrightarrow$ Breakdown of ordinary perturbation theory

- When do non–linear effects become important ?

$$D_\nu = \partial_\nu - igA_\nu$$

$$\boxed{\partial_\nu \sim gA_\nu \text{ when } \bar{A} \sim \bar{k}/g}$$

  $\Longrightarrow$ Strong–field regime of QCD

  Since A_ν is a fluctuating field, this condition should be understood in the sense of correlations.

- E.g.: Energy density $\sim \langle \mathrm{tr}\, F_{\mu\nu}^2 \rangle$

  Since: $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$,

  non-linear effects become important when:

  $\big\langle \big(\partial \cdot A\big)^2 \big\rangle \sim g^2 \big\langle A^4 \big\rangle$, that is: $\bar{k}^2 \big\langle A^2 \big\rangle \sim g^2 \big\langle A^2 \big\rangle^2$

$$\boxed{g^2 \big\langle A^2 \big\rangle \sim \bar{k}^2}$$

As we shall see:

- This happens for momenta k smaller than some intrinsic scale, characteristic of the density and of the interactions in the system.

- When this happens, gluon occupation numbers are large $(\sim 1/g^2)$, and a semi–classical description becomes possible.

$$\boxed{\text{Strong-field QCD} \longrightarrow \text{Classical effective theory}}$$

A. High–Temperature QCD

- $T > T_c \sim 170$ MeV $\Longrightarrow$ Quark–Gluon Plasma
- Thermal occupation number for gluons :

$$\frac{1}{V}\frac{\mathrm{d}N}{\mathrm{d}^3k} = n(\varepsilon_k) = \frac{1}{\mathrm{e}^{\beta\varepsilon_k}-1} \qquad (\beta = 1/T, \quad \varepsilon_k = \hbar k)$$

Since: $\frac{\mathrm{d}N}{\mathrm{d}k} \sim \frac{k^2}{\mathrm{e}^{\beta\hbar k}-1} \Longrightarrow$ Most gluons have $k \sim T$

When $k \sim T \quad \Longrightarrow \quad n(k) \sim 1$

But soft modes have large occupation numbers:

$k \ll T \quad \Longrightarrow \quad n(k) \simeq \frac{T}{\hbar k} \gg 1$

N.B. The same as the classical limit ($\hbar \to 0$).

- Average energy per soft mode:

$\varepsilon_k\, n(\varepsilon_k) \simeq T$ for $k \ll T \Longrightarrow$ classical 'equipartition'

Soft modes are quasi–classical !

- What is the typical strength of thermal fluctuations?

$$\bar{A} \equiv \sqrt{< A^2(t,\boldsymbol{x}) >}; \qquad < A^2 > \sim \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{n(\varepsilon_k)}{\varepsilon_k}$$

- Focus on fluctuations with size $\sim 1/\bar{k}$ $[\hbar = 1]$:

$$\langle A^2\rangle_{\bar{k}} \sim \int^{\bar{k}} \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{k}\,\frac{1}{\mathrm{e}^{\beta k}-1}$$

- If $\bar{k} \sim T \quad \Longrightarrow \quad \langle A^2 \rangle_T \sim T^2$

  $\Longrightarrow \; g^2 \langle A^2 \rangle_T \sim g^2 T^2 \; \ll \; \bar{k}^2 \sim T^2$

  Typical ($k \sim T$) fluctuations are perturbative.

- If $\bar{k} \ll T \quad \Longrightarrow \quad \langle A^2 \rangle_{\bar{k}} \sim \int^{\bar{k}} \frac{\mathrm{d}^3 k}{k} \frac{T}{k} \sim \bar{k}\, T$

  $g^2 \langle A^2 \rangle_{\bar{k}} \sim \bar{k}^2$ provided $\bar{k} \sim g^2 T$

  Fluctuations with $k \lesssim g^2 T$ are non–perturbative !

- Physical interpretation :

  Generation of a "gluon (screening) mass" $m_s \sim g^2 T$.

- For $k \leq m_s$: $\varepsilon_k \to \sqrt{k^2 + m_s^2} \simeq m_s$

  The occupation numbers "saturate" to $n \simeq 1/\alpha_s$:

$$n(k) \simeq \frac{T}{m_s} \sim \frac{1}{\alpha_s} \gg 1$$

- How to calculate in this non–perturbative regime ?

  a) Construct a classical effective theory for the soft modes ($k \sim g^2 T$) by integrating out the relatively hard modes ($k \sim T$ and gT) in perturbation th.

  b) Solve the classical eff. th. on a 3–dim. lattice

- References :

  F.Karsch, E.Laermann, hep-lat/0305025 (lattice)

  U. Kraemmer, A. Rebhan, hep-ph/0310337

  J.-P. Blaizot, E. Iancu, Phys. Rept. 359 (2002) 355

B. High–Energy QCD

- The "small–x" part of the wavefunction of a fast moving ($v \simeq c$) hadron
- High density of virtual partonic excitations (mostly gluons) which are "frozen" by Lorentz time dilation.
- Gluon occupation number :

$$n_g \;=\; \frac{(2\pi)^3}{2\cdot(N_c^2-1)}\,\frac{\mathrm{d}N_g}{\mathrm{d}^3k\,\mathrm{d}^3b}$$

(# of gluons of given spin and color per unit phase–space)

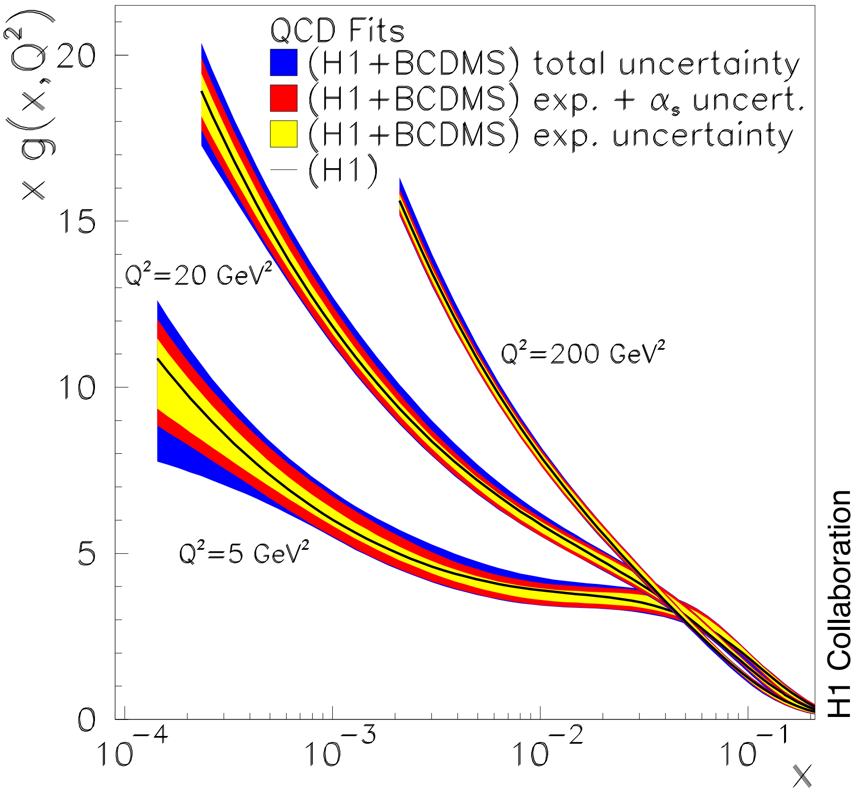
- n_g is not a priori known (unlike in the high–T case). But the gluon field fluctuations $\langle A^2\rangle$ can be directly extracted from the data !
- DIS $\Longrightarrow$ The gluons transverse size $\Delta x_\perp \sim 1/Q$ and their longitudinal momentum $k_z = xP$ are fixed by the measured quantities x and Q^2.
- The "gluon distribution" :

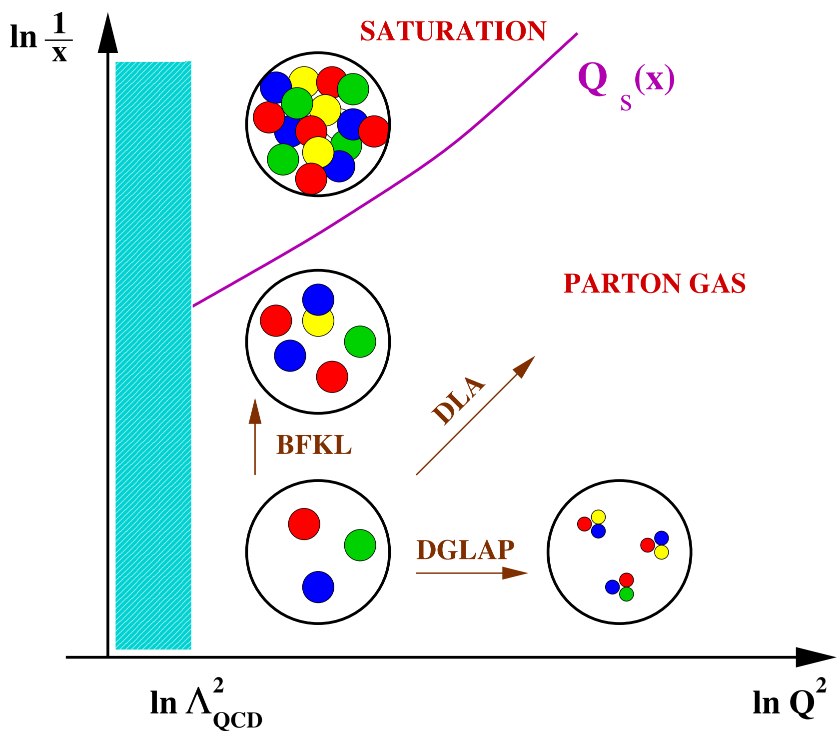
$$xG(x,Q^2) \;\equiv\; x\,\frac{\mathrm{d}N_g}{\mathrm{d}x} \;\approx\; \frac{\mathrm{d}N_g}{\mathrm{d}k_z\mathrm{d}b_z}$$

since: $\mathrm{d}k_z\mathrm{d}b_z \approx \mathrm{d}k_z/k_z = \mathrm{d}x/x$

$$\frac{xG(x,Q^2)}{\pi R^2} \;=\; \int^{Q^2}\frac{\mathrm{d}^2k_\perp}{(2\pi)^2}\,n_g \;=\; \langle A^2\rangle_Q$$

Q plays the same role as $\bar{k}$ in the general case.


QCD Fits
(H1+BCDMS) total uncertainty
(H1+BCDMS) exp. + α_s uncert.
(H1+BCDMS) exp. uncertainty
(H1)
$x\,g(x,Q^2)$
Q^2=20 GeV^2
Q^2=200 GeV^2
Q^2=5 GeV^2
x
H1 Collaboration


$\ln\frac{1}{x}$
SATURATION
$Q_s(x)$
PARTON GAS
DLA
BFKL
DGLAP
$\ln\Lambda^2_{QCD}$
$\ln Q^2$

- The condition for non–perturbative behaviour :

$$g^2 \langle A^2 \rangle_Q \;\sim\; Q^2$$

$\Longrightarrow$ An equation for $Q^2 = Q_s^2(x)$:

$$Q_s^2(x) \;\simeq\; \alpha_s \,\frac{xG(x, Q_s^2(\mathrm{x}))}{\pi R^2}$$

- Physical interpretation :

$$\boxed{Q_s(x) \;=\; \text{“saturation momentum”}}$$

For $k_\perp \,\leq\, Q_s(x)$: strong non–linear effects which should lead to gluon saturation :

$$n_g \sim \frac{1}{\alpha_s} \;\gg 1 \qquad \text{for} \quad k_\perp \,\leq\, Q_s(x)$$

- $Q_s(x)$ increases rapidly with the energy $(1/x)$, and also with the atomic number A for a large nucleus.
- For protons at HERA, or heavy nuclei at RHIC, Q_s is estimated to be on the order of 1 GeV
- Saturated gluons $\Longleftrightarrow$ Strong classical color fields

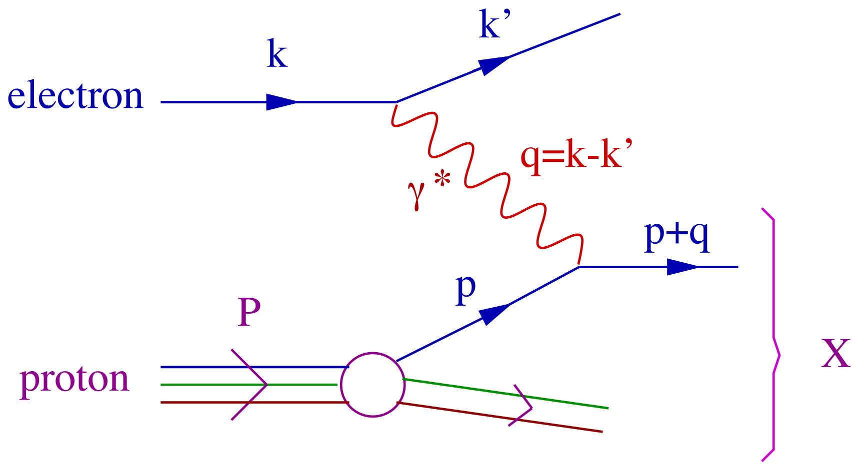
$$\boxed{\text{Color Glass Condensate}}$$

II. Deep Inelastic Scattering at Small–x

A. Generalities (kinematics, structure functions, parton picture)

electron(k) + proton(P) $\longrightarrow$ electron(k') + X(P_X)

- X : undetected hadronic system



- $s = (P+k)^2$: center–of–mass energy squared
- Two independent kinematical invariants :

1) $$Q^2 \equiv -q^\mu q_\mu = -(k-k')^2 \qquad (Q^2 \geq 0)$$

2) $$x \equiv \frac{Q^2}{2P\cdot q} = \frac{Q^2}{Q^2+W^2-M^2} \qquad (\text{Bjorken's } x)$$

Here: $W^2 \equiv (P+q)^2 \geq M^2 \Longrightarrow 0 \leq x \leq 1$

- For $Q^2 \ll M_Z^2 \Longrightarrow$ one photon exchange (γ^*)

- "Deep inelastic" or Bjorken limit :

$$\boxed{\text{Both } Q^2 \text{ and } P\cdot q \gg M^2 \text{ with } x = \text{fixed}}$$

- "Small-x" or High energy limit :

$$\boxed{W^2 \gg Q^2 > M^2 \quad \Longrightarrow \quad x \simeq \tfrac{Q^2}{W^2} \ll 1}$$

- DIS cross–section $\Longrightarrow$ Two "structure functions"

  The total cross section for virtual photon absorbtion:

$$\sigma_{\gamma^* p}(x, Q^2) \;=\; \sigma_T + \sigma_L \;=\; \frac{4\pi^2\alpha_{\rm em}}{Q^2}\, F_2(x, Q^2)$$

  $\sigma_{T,L} \;=\; (4\pi^2\alpha_{\rm em}/Q^2)F_{T,L},$

  $F_2 = F_T + F_L, \quad F_1 = F_T/2x$

- Kinematics + (Lorentz and gauge) symmetries
- What is the physical interpretation of $F_{T,L}$?

  A measure of the quark and gluon distributions

  in the proton in a suitable frame and gauge.

- The proton: A loosely bound assemblage of

  constituents called 'partons' (quarks and gluons)

  Why "loosely" ? Because of asymptotic freedom:

$$\alpha_s(Q^2) \;\simeq\; \frac{1}{b\ln(Q^2/\Lambda^2_{\rm QCD})} \ll 1 \quad \text{for} \quad Q^2 \gg \Lambda^2_{\rm QCD}$$

  $b = (11N_c - 2N_f)/12\pi,\ N_c = 3,\ \Lambda_{\rm QCD} \sim 200$ MeV.

  $\Longrightarrow$ Typical binding energy $\sim \Lambda_{\rm QCD}$

- A parton: A virtual excitation with virtuality $p^2 - m^2 \sim \Lambda^2_{\rm QCD}$ and lifetime $\tau_{\rm part} \sim 1/\Delta p_0$
- Proton Infinite Momentum Frame (IMF):

  Any frame in which $P_z \gg M$:

  $P^\mu \simeq (P + \frac{M^2}{2P}, 0, 0, P) \simeq (P, 0, 0, P)$
- In IMF, partons are longlived and nearly collinear.

  Indeed: $p^\mu = (p_0, p_\perp, p_z)$ with $p_z = \xi P$ and $0 \le \xi \le 1$

  ξ = longitudinal momentum fraction of the parton

  Typically: $p_\perp \sim \Lambda_{\rm QCD}$ and $p^2 - m^2 \sim \Lambda^2_{\rm QCD}$

  that is: $p_0^2 - E_p^2 \sim \Lambda^2_{\rm QCD}$ with $E_p = \sqrt{m^2 + p_\perp^2 + p_z^2}$

  But: $p_z = \xi P \gg m, \Lambda_{\rm QCD} \Longrightarrow E_p \simeq p_z \gg p_\perp$

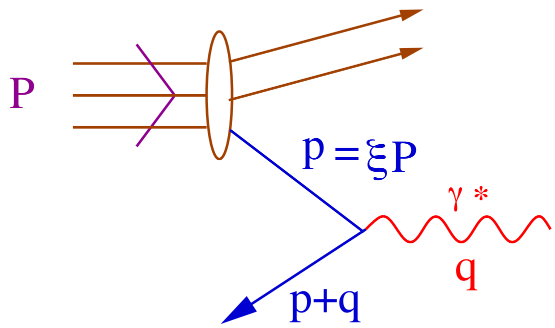
  Also: $p_0 \simeq E_p \simeq p_z$ and $\Delta p_0 \sim p_0 - p_z \simeq \Lambda^2_{\rm QCD}/2p_z$

  $$p^\mu \approx \xi(P, 0_\perp, P) = \xi P^\mu$$

  $$\tau_{\rm part} \sim \frac{2p_z}{\Lambda^2_{\rm QCD}} \gg \tau_{\rm vac} \sim \frac{1}{p_z}$$

  In this frame, the parton is well separated from the vacuum fluctuations.
- DIS in IMF: γ^* is absorbed by individual quarks

This is best seen in the Breit frame ($q_z = 0$)



$P^\mu = (P\ ,0_\perp\ ,P)$

$q^\mu = (q_0\ ,q_\perp\ ,0)$

i) Transverse resolution :

$q_0 = \frac{P \cdot q}{P} \to 0$ as $P \to \infty \Longrightarrow Q^2 \equiv -q_0^2 + q_\perp^2 \simeq q_\perp^2$

$\Longrightarrow$ The γ^* momentum is mainly transverse

γ^* is absorbed over a transverse distance $\Delta x_\perp \sim 1/Q$

Simultaneous scattering off two or more quarks is suppressed by powers of $1/Q^2$ ("higher–twist").

ii) Duration of the scattering process:

$$\tau \sim \frac{1}{E_p + q_0 - E_{p+q}} \simeq \frac{2P\xi}{Q^2} \ll \tau_{\rm part} \sim \frac{2P\xi}{\Lambda^2}$$

Indeed: $E_p \simeq p_z = \xi P$, $q_0 \approx 0$, and

$$E_{p+q} = \sqrt{m^2 + q_\perp^2 + p_z^2} \simeq p_z + Q^2/2p_z$$

γ^* is absorbed very fast compared to natural time scales

The proton constituents move almost freely over the very short times scales corresponding to DIS !

iii) Mass–shell constraint for the outgoing quark :

$$0 = (p+q)^2 = q^2+2p\cdot q = -Q^2+2\xi P\cdot q = -2P\cdot q(x-\xi)$$

$$\boxed{\xi = x}$$

- To compute $\sigma_{\gamma^* p}$, it is enough to know the quark "distribution" at the time of scattering.
- Parton model (Bjorken, 69; Feynman, 72)
  "Partons": strictly on–shell and collinear: $p^\mu = \xi P^\mu$.
  (Consistent with QCD to lowest order in α_s.)
- The quark distribution is fully specified as :

  $q(\xi)\mathrm{d}\xi \;=\;$ the probability of finding a quark with longitudinal fraction between ξ and $\xi + \mathrm{d}\xi$
- Parton model calculation of the $\gamma^* p$ cross-section

$$\sigma_{\gamma^* p} \;=\; \sum_f \int_0^1 \mathrm{d}\xi \,\left[q_f(\xi) + \bar{q}_f(\xi)\right] \hat{\sigma}_{\gamma^* f}$$

- Partonic cross-sections :

$$\hat{\sigma}_T \;=\; \frac{4\pi^2\alpha_{\rm em}}{Q^2}\, e_f^2\, \xi\, \delta(\xi - x), \qquad \hat{\sigma}_L = 0$$

- Parton model predictions for the structure functions:

$$\boxed{F_2(x) \;=\; \sum_f e_f^2 \left[x q_f(x) + x\bar{q}_f(x)\right] \;=\; F_T(x)}$$

— $F_2(x)$ is independent of Q^2 ("Bjorken scaling")
— $F_L = 0$ ("Callan–Gross relation"): spin 1/2.

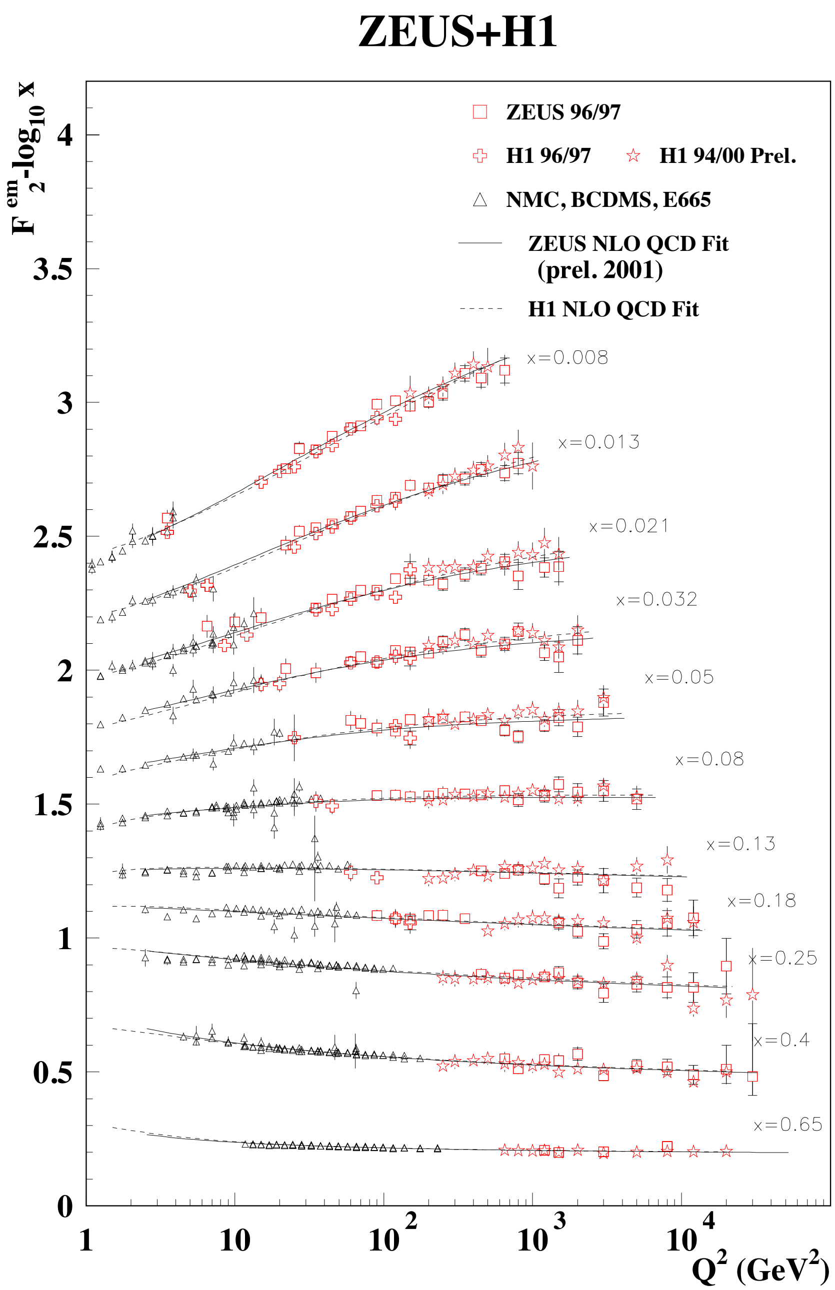


Figure 1: The F_2 structure function as measured by the H1 and ZEUS experiments for bins at high x as a function of Q^2. The bins centred around $x = 0.25$ are where scaling was originally observed in the SLAC experiments.

From the review paper "Lectures on HERA physics", by B. Foster, EPJdirect A1, 1–11 (2003); hep-ex/0206011.

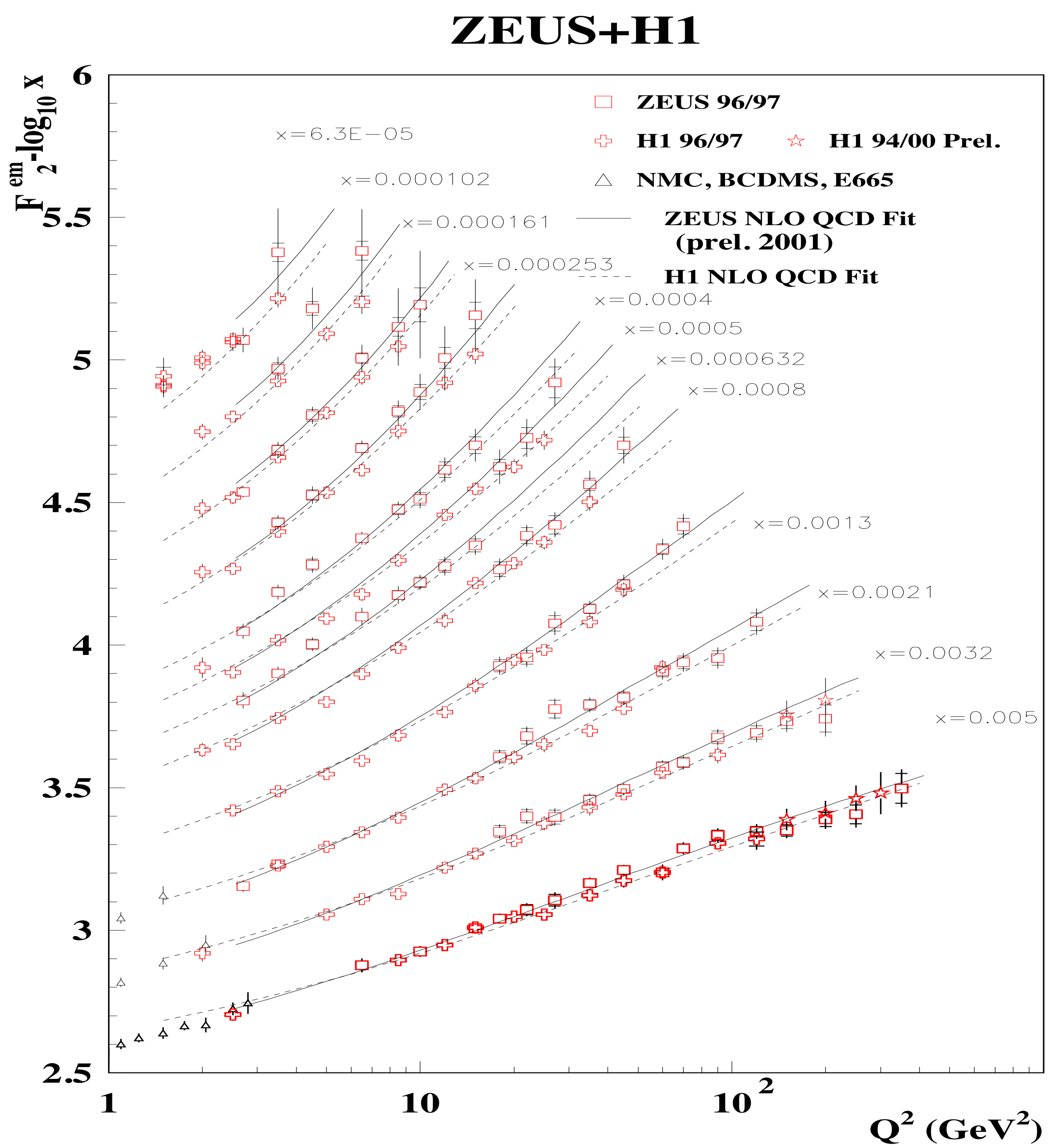
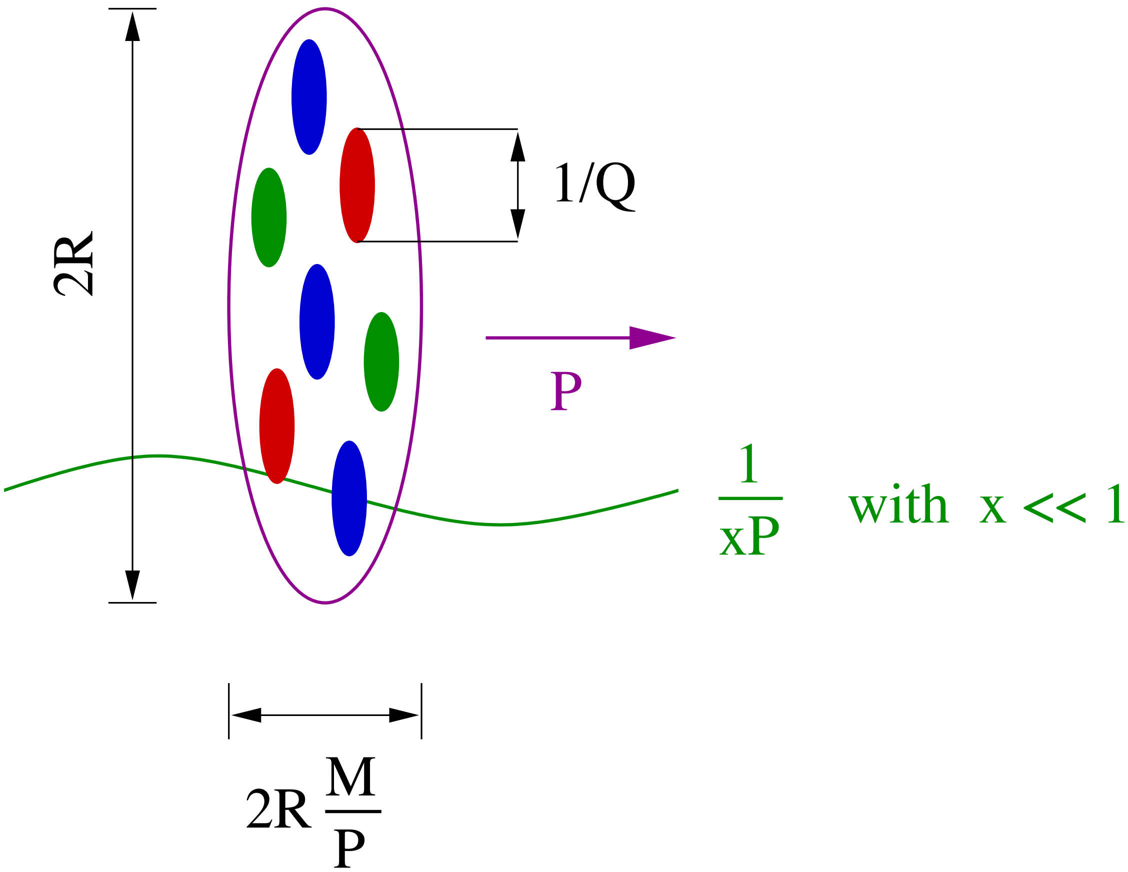


Figure 2: The F_2 structure function as measured by the H1 and ZEUS experiments for bins at low x as a function of Q^2. The violations of Bjorken scaling are now obvious.

From the review paper "Lectures on HERA physics", by B. Foster, EPJdirect A1, 1–11 (2003); hep-ex/0206011.

- A pictorial view of the hadron as “seen” in DIS and in IMF :



- Lorentz contraction in the longitudinal direction with $\gamma = P/M \gg 1$
- Uncertainty principle: $\Delta z \sim \frac{1}{p_z} = \frac{1}{xP}$
- Partons with typical x (not too small) are confined to the Lorentz contracted disk.
- Partons with very small $x \ll 1$ (“wee”, or “slow”) are NOT !

- Small–$x \longleftrightarrow$ High energy

$$x = \frac{Q^2}{Q^2 + W^2} \simeq \frac{Q^2}{W^2} \ll 1 \quad \text{when} \quad W^2 \gg Q^2$$

- With increasing Q^2 : one increases the spatial resolution in the transverse plane : $\Delta x_\perp \sim 1/Q$

- With decreasing x :

  — one increases the resolution in time: $\Delta t \sim \frac{2xP}{Q^2}$

  — but one reduces the spatial resolution in the longitudinal direction: $\Delta z \sim \frac{1}{xP}$

- Partons are not free particles, but d.o.f. of QCD !

  What should be the effects of their interactions ?

  — give the partons a transverse dimension;

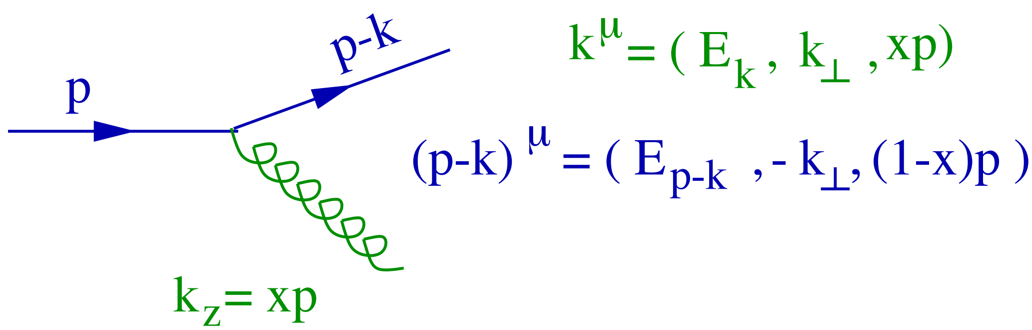
  — introduce gluon influence on F_2;

  — generate a dense gluonic system at very small–x;

  — breakdown of the "leading–twist" approximation in the gluonic sector.

B. Parton Evolution in QCD

- In QCD, partons have a substructure, in terms of virtual fluctuations.
- Parton cascades : A fast parton ($p^\mu \simeq (p, 0_\perp, p)$) can decay into two partons which are still fast, and thus will exist for a long time:

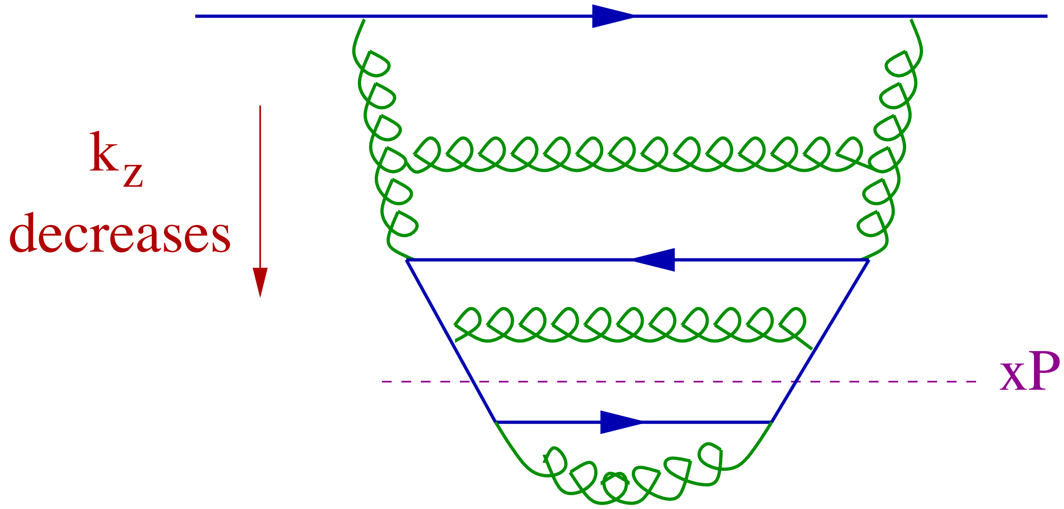


$$\Delta E = -p + \sqrt{x^2p^2 + k_\perp^2} + \sqrt{(1-x)^2p^2 + k_\perp^2} \approx \frac{k_\perp^2}{2xp} + \frac{k_\perp^2}{2(1-x)p}$$

$$\Delta t \sim \frac{\min(x, 1-x)p}{k_\perp^2} \gg \tau_{\rm vac} \sim \frac{1}{k_z}$$

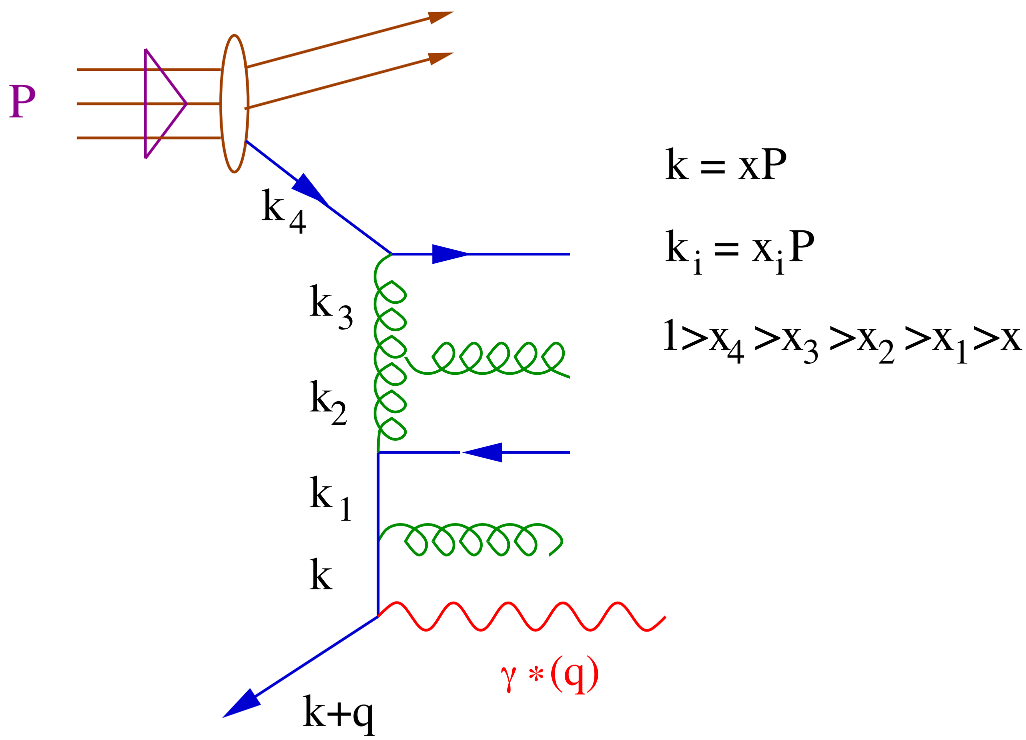
- The ensuing partons can split again, into fluctuations with even smaller k_z, and thus shorter lifetimes.
- In the absence of any external interaction, the cascade develops until very slow partons, with $k_z = xp \sim k_\perp$, get formed.

  Then, the partons recombine back.
- Partons within the same cascade are coherent.



- In DIS, γ^* hooks a quark having $k_z = xP$ and transverse size $\Delta x_\perp \sim 1/Q$ in some cascade.

  The coherence of that cascade is lost, and its partons are released in the final state.

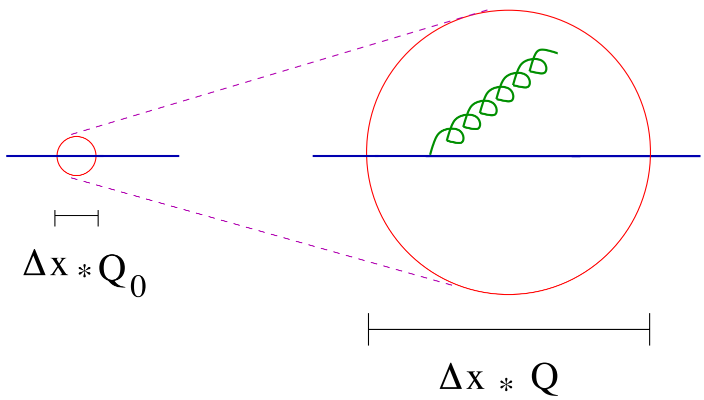


$$F_2(x,Q^2) = \sum_f e_f^2 \left[xq_f(x,Q^2) + x\bar{q}_f(x,Q^2)\right]$$

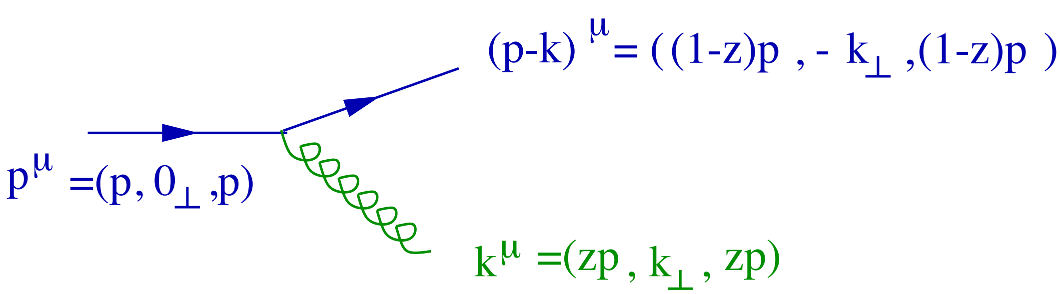
$q_f(x,Q^2)\mathrm{d}x$ = # of quarks of flavor f with longitudinal fraction between x and $x+\mathrm{d}x$ and localized in transverse space to a size $1/Q$

$$xq_f(x,Q^2) = \int^{Q^2} \mathrm{d}^2k_\perp \, x \, \frac{\mathrm{d}N_f}{\mathrm{d}x\mathrm{d}^2k_\perp}$$

- With increasing Q^2, one can "see" components of the quark which are smaller and smaller.



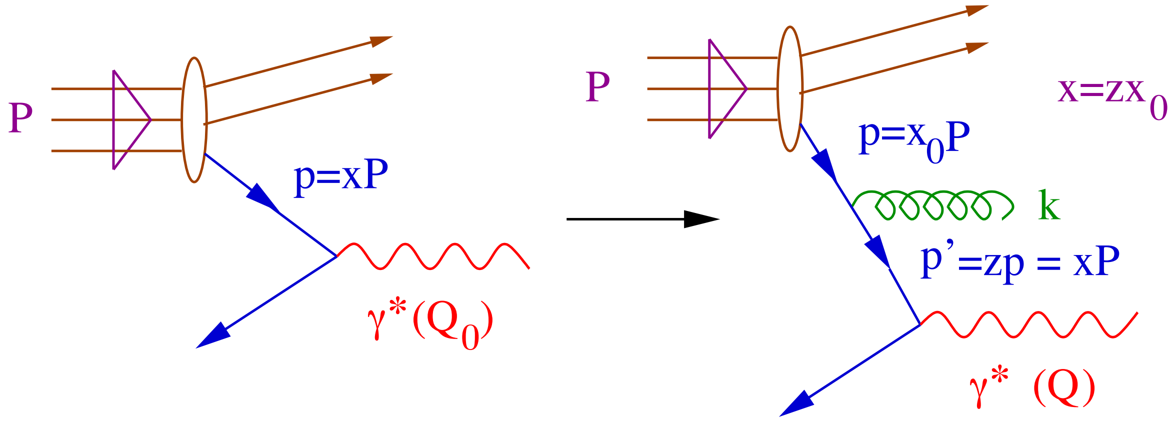
- Nearly collinear splitting ($k_\perp \ll k_z = zp$)



$$\mathrm{d}\mathcal{P} = \frac{\alpha_s C_F}{2\pi}\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\frac{1+(1-z)^2}{z}\mathrm{d}z \quad (C_F = t^a t^a = 4/3)$$

Collinear evolution of the quark distribution

- Increase $Q_0^2 \longrightarrow Q^2$ at fixed x :



$$q_f(x,Q^2) = q_f(x,Q_0^2) + \frac{\alpha_s C_F}{2\pi} \ln \frac{Q^2}{Q_0^2} \int_x^1 \frac{\mathrm{d}z}{z} \frac{1+z^2}{1-z} q_f\left(\frac{x}{z}, Q_0^2\right)$$

where the integral over z has been generated as:

$$\int_0^1 \mathrm{d}x_0\, q_f(x_0) \int_0^1 \mathrm{d}z\, \delta(x - zx_0) \;=\; \int_x^1 \frac{\mathrm{d}z}{z} q_f\left(\frac{x}{z}\right)$$

- If $\alpha_s \ln(Q^2/Q_0^2) \sim 1 \Longrightarrow$ need for resummation :

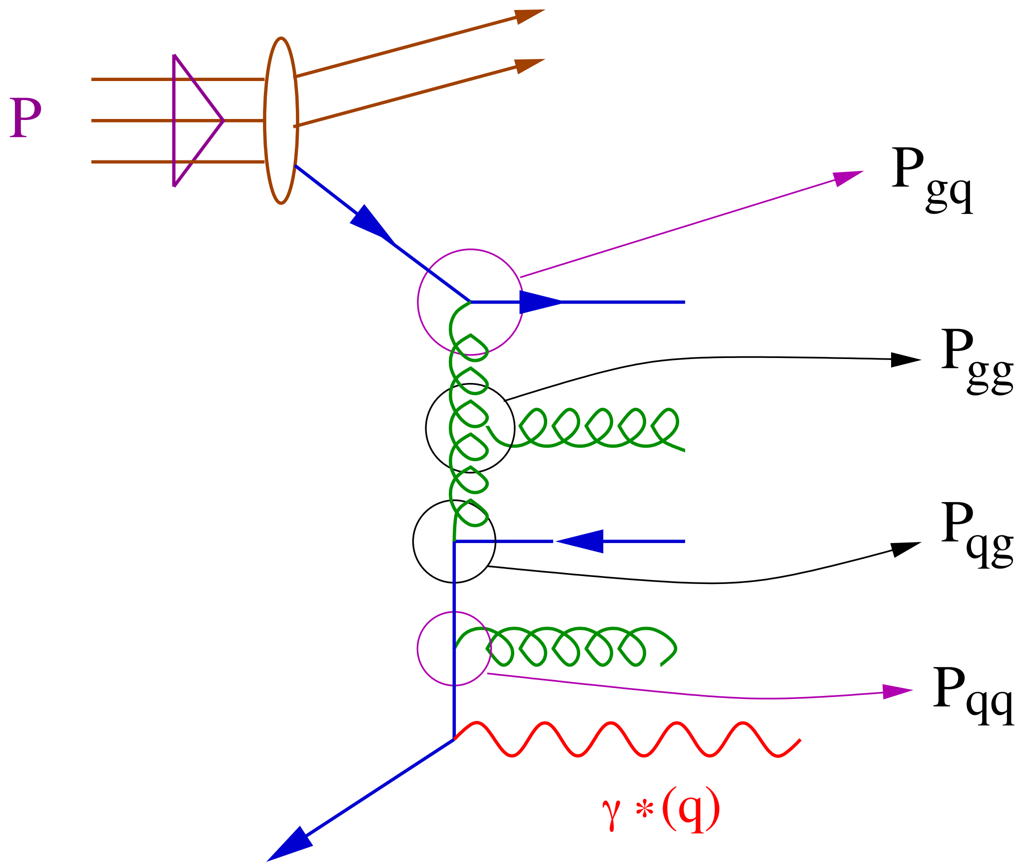
$$\frac{\partial q_f}{\partial \ln Q^2}(x,Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} P_{qq}(z)\, q_f\left(\frac{x}{z}, Q^2\right)$$

with the $q \to q$ splitting function :

$$P_{qq}(z) \;=\; C_F \left\{ \frac{1+z^2}{(1-z)_+} + \underbrace{\frac{3}{2}\delta(1-z)}_{\text{virtual corr.}} \right\}$$

- All orders $(\alpha_s \ln Q^2)^n$
- Local in Q^2, non–local in x

Adding the gluons



- Coupled equations for $q_f(x,Q^2)$ and $G(x,Q^2)$:

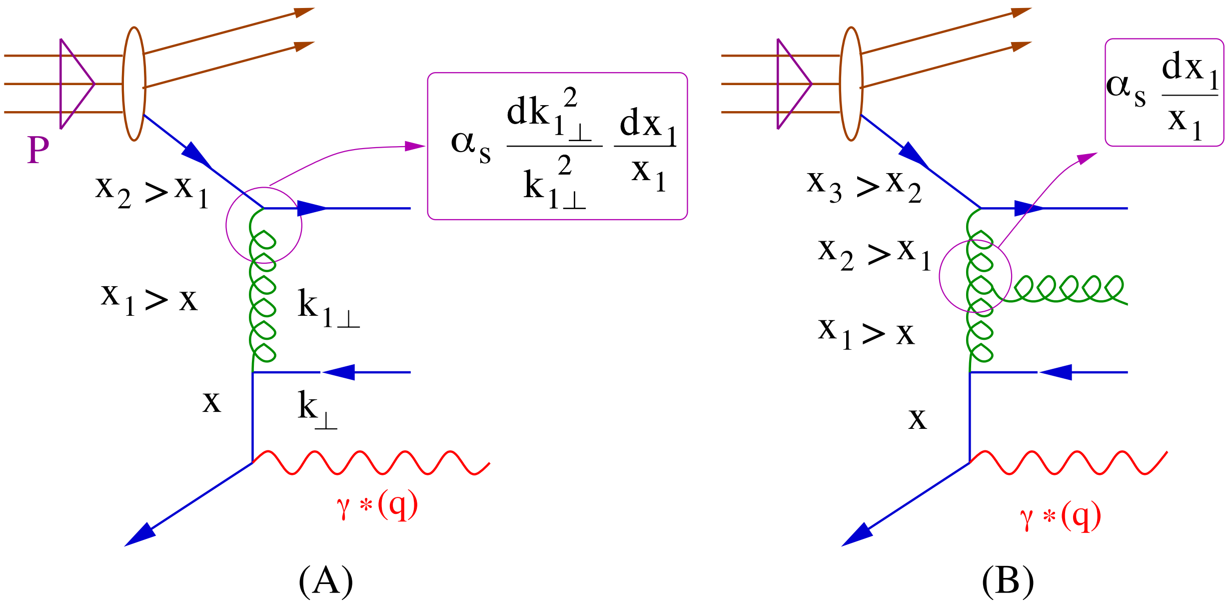
$$\frac{\partial q_f(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left\{P_{qq}(z)\, q_f\left(\frac{x}{z},Q^2\right) + \right.$$
$$\left. + P_{qg}(z)\, G\left(\frac{x}{z},Q^2\right)\right\}$$

$$\frac{\partial\, G(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left\{P_{gg}(z) G\left(\frac{x}{z},Q^2\right) + \right.$$
$$\left. + P_{gq}(z) \sum_{f,\bar{f}} q_f\left(\frac{x}{z},Q^2\right)\right\}$$

DGLAP (Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)

The small–x limit of DGLAP evolution

- Gluon splitting dominates at small x !



(A) $\alpha_s \int_x^{x_2} \frac{dx_1}{x_1} = \alpha_s \ln \frac{x_2}{x} \simeq \alpha_s \ln(1/x)$

(B) $\alpha_s^2 \int_x^{x_3} \frac{dx_2}{x_2} \int_x^{x_2} \frac{dx_1}{x_1} \simeq \frac{1}{2}\big(\alpha_s \ln(1/x)\big)^2$

- Is this effect included in DGLAP ?

  Only to the extent that $Q^2 \gg k_\perp^2 \gg k_{1\perp}^2 \gg k_{2\perp}^2 \dots$!

  (A) $\alpha_s \ln(1/x) \ln Q^2$; (B) $\left[\frac{1}{2!}\, \alpha_s \ln(1/x) \ln Q^2\right]^2$

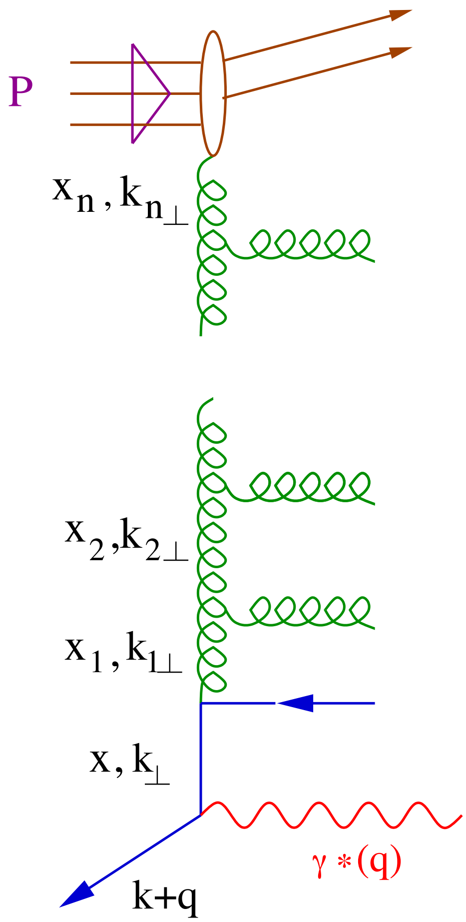
- These are the terms which dominate when <u>both</u> $\ln(1/x)$ and $\ln Q^2$ are large, such that:

$$\boxed{\alpha_s \ln(1/x) \ln Q^2 \sim 1}$$

but $\alpha_s \ln(1/x) \ll 1$ and $\alpha_s \ln Q^2 \ll 1$

"Double leading-log approximation" (DLA)

The DLA evolution equation



$$Q^2 \gg k_\perp^2 \gg k_{1\perp}^2 \gg \; \ldots \gg k_{n\perp}^2$$

$$x \ll x_1 \ll x_2 \ll \cdots \ll x_n \ll 1$$

Resummation of all terms

$$\frac{1}{(n!)^2}\left(\alpha_s \ln Q^2 \ln \frac{1}{x}\right)^n$$

$$\frac{\partial q_f}{\partial \ln Q^2}(x, Q^2) \simeq \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}z}{z} P_{qg}(z)\, G\left(\frac{x}{z}, Q^2\right)$$

$$\frac{\partial\, xG}{\partial \ln Q^2}(x, Q^2) \simeq \frac{\alpha_s N_c}{\pi}\int_x^1 \frac{\mathrm{d}z}{z}\,\frac{x}{z}\, G\left(\frac{x}{z}, Q^2\right)$$

Exercice Show that in the DLA:

$$\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} \simeq \frac{\alpha_s}{3\pi}\left(\sum_f e_f^2\right) xG(x, Q^2)$$

The evolution is driven by the gluon distribution !

- Local evolution equation for $xG(x,Q^2)$

$$\frac{\partial\, xG(x,Q^2)}{\partial \ln Q^2 \partial\, \ln \frac{1}{x}} \;=\; \frac{\alpha_s N_c}{\pi}\, xG(x,Q^2)$$

Exercice

Show that asymptotically ($\alpha_s \ln(1/x) \ln Q^2 \gg 1$) the solution $xG(x,Q^2)$ has the following behaviour:

i) For fixed coupling α_s (with $\bar{\alpha}_s \equiv \alpha_s N_c/\pi$):

$$xG(x,Q^2) \;\propto\; \exp\left\{2\sqrt{\bar{\alpha}_s\, \ln\frac{1}{x}\, \ln\frac{Q^2}{Q_0^2}}\right\}$$

where Q_0^2 is an arbitrary reference scale.

ii) For running coupling $\alpha_s(Q^2) = \frac{1}{b\ln(Q^2/\Lambda^2)}$:

$$xG(x,Q^2) \;\propto\; \exp\left\{2\sqrt{\frac{N_c}{\pi b}\, \ln\frac{1}{x}\, \ln\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}}\right\}$$

- Quite rapid growth with $1/x$: slower than any power of $1/x$, but faster than any power of $\ln(1/x)$.
- But DLA is not truly interesting:

— Irrelevant for HERA: the small–x data correspond to rather low Q^2, with $\alpha_s \ln(1/x) \gtrsim 1$ and $\ln(1/x) > \ln Q^2$.

— DLA is not the evolution towards high density partonic systems: An increasing number of smaller and smaller gluons which do not overlap with each other.