Statistical description of the initial state fluctuations and mode-by-mode dynamical evolution

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Strong-interaction matter under extreme conditions

QCD challenges from pp to AA collisions, Münster, September 2-6, 2024

Statistical description of the initial state fluctuations & mode-by-mode dynamical evolution

- Statistical description of the initial state fluctuations
  - Generic idea: average state and fluctuation modes
  - Examples of application
- Mode-by-mode dynamical evolution
- Outlook

N.B., M.Borrell, N.Feld, H.Roch, S.Schlichting, C.Werthmann, PRC **107** (2023) 034905

N.B., H.Roch, A.Schütte, arXiv:2402.07888

R.Krupczak, N.B., H.Roch, in preparation

Thanks to Renata for many plots!

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Statistical description of the initial state fluctuations

- Take your favorite model, with event-by-event fluctuations, for the initial state of a nuclear collision:
  - MCGlauber [used in PRC **107** (2023) 034905 & work(s) in preparation]
  - saturation-based [used in PRC **107** (2023) 034905]
  - TRENTo, IP-Glasma, EKRT, Jazma, McDIPPER...
  - toy model with independent hot spots [used in arXiv:2402.07888].

  ☞ In principle 2D or 3D, energy or entropy density (or whole $T^{\mu\nu}$), with or without conserved charges.
  Examples shown in the following are 2D-energy density profiles, mostly from a nucleon-based MCGlauber.

🤔 What about dynamical initializations [SMASH, models with excited strings]?

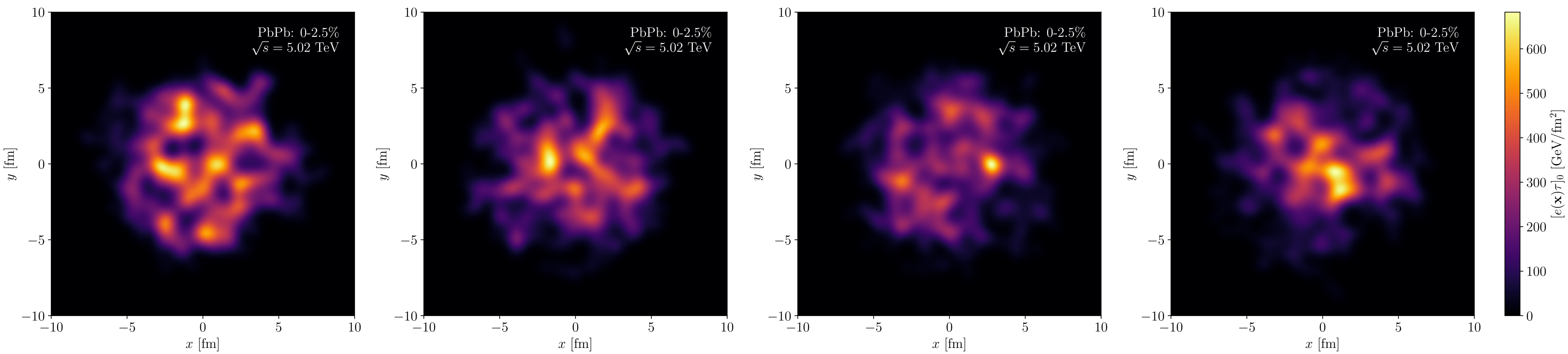
Statistical description of the initial state fluctuations

- Take your favorite model, with event-by-event fluctuations, for the initial state of a nuclear collision
- and generate a (large) set of initial states $\{\Phi^{(i)}(\mathbf{x})\}$ "under the same conditions"
  - same colliding system at fixed collision energy!

    ☞ hereafter, mostly Pb-Pb @ 5.02 TeV
  - collisions at fixed impact parameter [PRC **107** (2023) 034905]
  - or within a given centrality class [in prep.]
  - or with a fixed average geometry [arXiv:2402.07888].

Statistical description of the initial state fluctuations

- Take your favorite model, with event-by-event fluctuations, for the initial state of a nuclear collision
- and generate a (large) set of initial states $\{\Phi^{(i)}(\mathbf{x})\}$ "under the same conditions" ☞ $N_{\rm ev} = 2^{21}$ per run in our simulations.



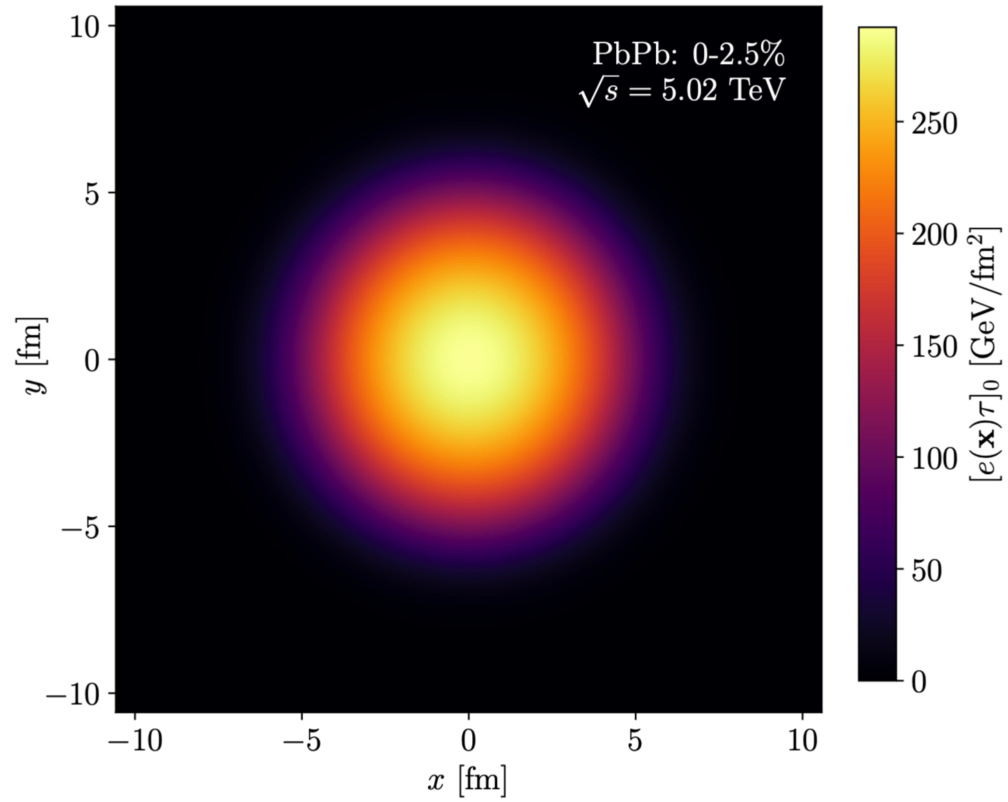
...

Here: transverse plane discretized on a grid with $N_{\rm s} \times N_{\rm s} = 192 \times 192$ points, with a grid spacing of ≈ 0.11 fm.

Statistical description of the initial state fluctuations

From the $N_{\rm ev}$ initial states $\{\Phi^{(i)}(\mathbf{x})\}$

- compute the "average initial state" $\bar{\Psi}(\mathbf{x}) \equiv \frac{1}{N_{\rm ev}} \sum_i \Phi^{(i)}(\mathbf{x})$



☞ (Almost) rotationally symmetric.

R.Krupczak, N.B., H.Roch, in prep.

Statistical description of the initial state fluctuations

From the $N_{\rm ev}$ initial states $\{\Phi^{(i)}(\mathbf{x})\}$

- compute the "average initial state" $\bar{\Psi}(\mathbf{x}) \equiv \frac{1}{N_{\rm ev}} \sum_i \Phi^{(i)}(\mathbf{x})$

☞ A random initial state may be seen as a random fluctuation about this average state:

$$\Phi^{(i)}(\mathbf{x}) = \bar{\Psi}(\mathbf{x}) + \delta\Phi^{(i)}(\mathbf{x})$$

The goal is now to characterize the fluctuating parts $\{\delta\Phi^{(i)}(\mathbf{x})\}$.

Statistical description of the initial state fluctuations

From the $N_{\rm ev}$ initial states $\{\Phi^{(i)}(\mathbf{x})\}$

- compute the "average initial state" $\bar{\Psi}(\mathbf{x}) \equiv \frac{1}{N_{\rm ev}}\sum_i \Phi^{(i)}(\mathbf{x})$
- and a basis of **unnormalized**, orthogonal "fluctuation modes" $\{\Psi_l(\mathbf{x})\}$ such that the expansion coefficients $\{c_l^{(i)}\}$ of the fluctuations

$$\Phi^{(i)}(\mathbf{x}) = \bar{\Psi}(\mathbf{x}) + \delta\Phi^{(i)}(\mathbf{x}) = \bar{\Psi}(\mathbf{x}) + \sum_l c_l^{(i)}\Psi_l(\mathbf{x})$$

satisfy $\left\langle c_l^{(i)}\right\rangle = 0$ and $\left\langle c_l^{(i)} c_m^{(i)}\right\rangle = \delta_{lm}$

where $\langle\cdots\rangle$ denotes the average over initial states.

Note: $\bar{\Psi}(\mathbf{x})$ is **not** one of the basis vectors.

Determining the fluctuation modes

One can show [see PRC **107** (2023) 034905] that they are eigenvectors to the autocorrelation function of the initial-state fluctuations:

$$\rho(\mathbf{x},\mathbf{y}) \equiv \frac{1}{N_{\rm ev}}\sum_i \delta\Phi^{(i)}(\mathbf{x})\,\delta\Phi^{(i)}(\mathbf{y})$$
$$= \frac{1}{N_{\rm ev}}\sum_i \Phi^{(i)}(\mathbf{x})\,\Phi^{(i)}(\mathbf{y}) - \bar{\Psi}(\mathbf{x})\,\bar{\Psi}(\mathbf{y})$$

☞ Numerically, one has to diagonalize a big matrix:

$$(N_{\rm fields}\cdot N_{\rm points}) \times (N_{\rm fields}\cdot N_{\rm points})$$

In our calculations $N_{\rm fields}$ = 1 (energy density), $N_{\rm points}$ = 192^2 (2D grid), so we diagonalize a $192^2 \times 192^2$-matrix: $\mathcal{O}$(1 day) (for each run).

⋮

But when it's done, you get nice plots for your paper / thesis!

Determining the fluctuation modes

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☞ Numerically, one has to diagonalize a big matrix:

- eigenvectors ⟿ fluctuation modes $\{\Psi_l(\mathbf{x})\}$
- eigenvalues λ_l ⟿ strength / importance of the modes

From now on, the subscript l reflects the mode strength, i.e. the $\{\Psi_l\}$ are sorted by decreasing λ_l.

Remark on the normalization of the fluctuation modes

- Slide 7: $\Phi^{(i)}(\mathbf{x}) = \bar{\Psi}(\mathbf{x}) + \sum_l c_l^{(i)} \Psi_l(\mathbf{x})$ with $\langle c_l^{(i)} c_m^{(i)} \rangle = \delta_{lm}$

  ☞ The amplitude of the fluctuations of the expansion coefficients c_l is (arbitrarily) fixed to unity for all modes [see slide 16].

  But $\Phi^{(i)}(\mathbf{x})$ has a physical dimension (e.g.: energy density).

  $\Rightarrow$ The $\{\Psi_l(\mathbf{x})\}$ must carry that dim., and cannot be normalized to 1!

- Slide 9: eigenvalues λ_l ⟿ strength / importance of the modes

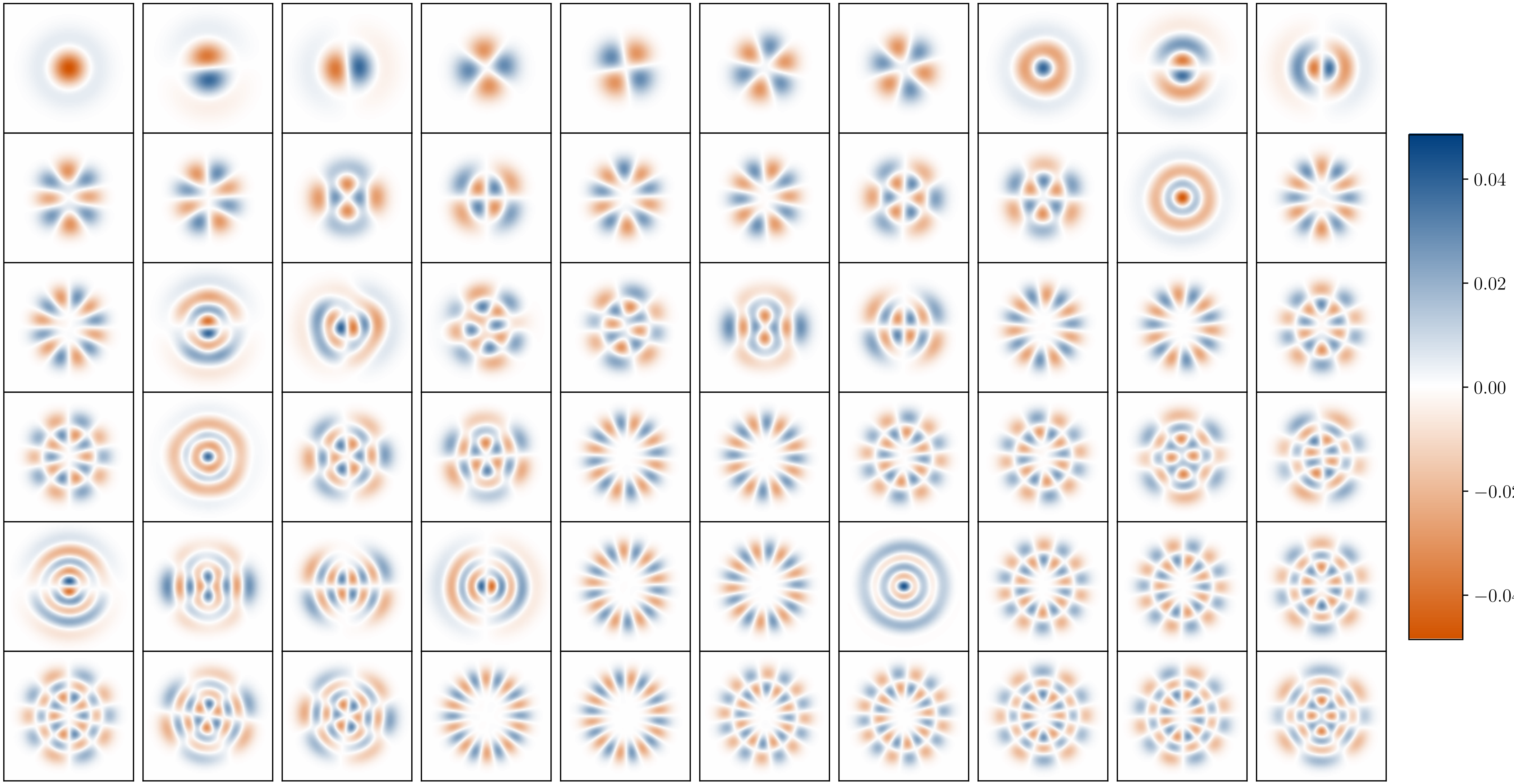
  ☞ Actually, one has $\lambda_l = \int [\Psi_l(\mathbf{x})]^2 \, \mathrm{d}^2\mathbf{x} \equiv \|\Psi_l\|^2$

If you prefer, you may work with normalized fluctuation modes, yet in that case the fluctuations of the c_l cannot be unity.

Eigenvectors (normalized)

Pb-Pb at 5.02 TeV, 0-2.5% centrality

(nucleon-based MCGlauber, fixed impact-parameter direction)

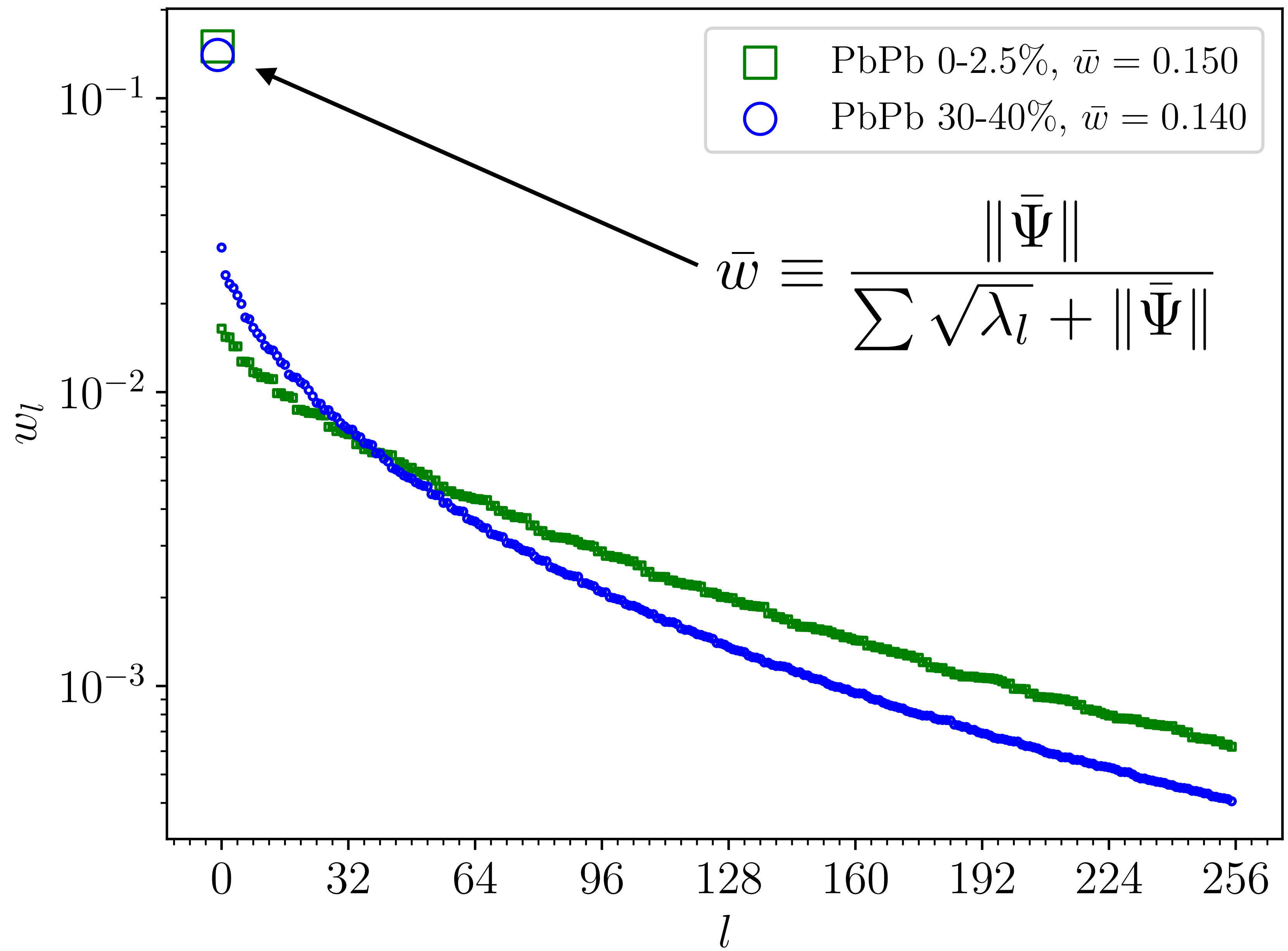


R.Krupczak, N.B., H.Roch, in prep.

Eigenvalues

Rather, "relative weights" $w_l \equiv \dfrac{\sqrt{\lambda_l}}{\sum\sqrt{\lambda_l}+\|\bar{\Psi}\|}$ with $\|\bar{\Psi}\|^2 \equiv \int \left[\bar{\Psi}(\mathbf{x})\right]^2 \mathrm{d}^2\mathbf{x}$

and $\sqrt{\lambda_l} = \|\Psi_l\|$



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Eigenvalues / relative weights

Is there any information in the slope of the spectrum? Yes there is!

Use (toy) 2D-model* with independent sources ("hot spots") distributed randomly according to predetermined distribution (here: Gaussian).

"Independent Hot Spot Model"

One can then compare the spectra obtained in runs with:

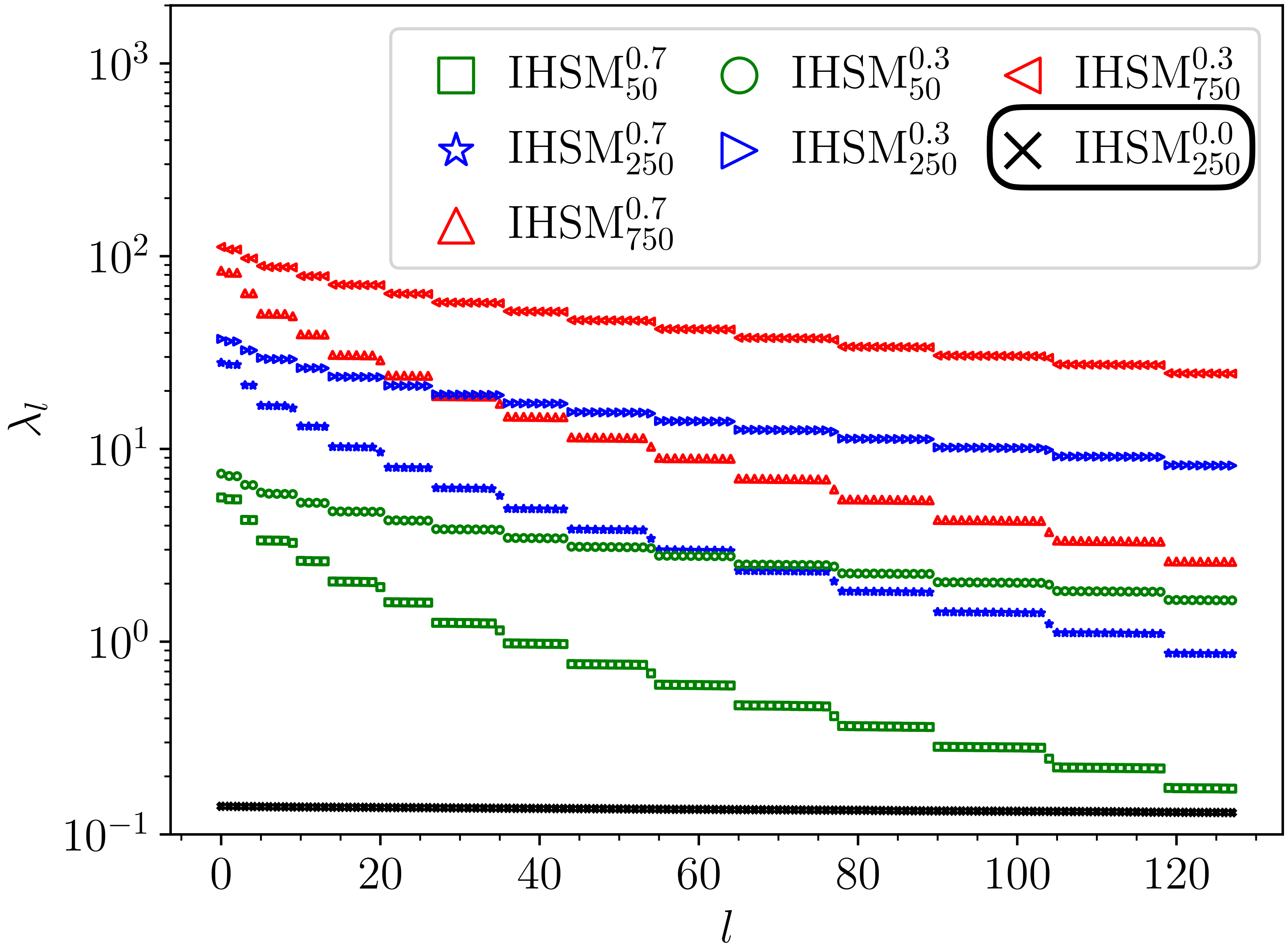
- different (fixed) numbers of hot spots $N_{\rm src}$
- different (fixed) source sizes $\sigma_{\rm src}$
- and more (different source weights, fluctuating $N_{\rm src}$ or $\sigma_{\rm src}$... not shown).

In the next two slides, runs labeled $\text{IHSM}^{\sigma_{\rm src}}_{N_{\rm src}}$ with $\sigma_{\rm src}$ in fm.

* long history: Bhalerao & Ollitrault, PLB **641** (2006) 260; Bhalerao, Luzum & Ollitrault, PRC **84** (2011) 024910; Bzdak, Bożek & McLerran, NPA **927** (2014) 15; Başar & Teaney, PRC **90** (2014) 054903; Bzdak & Skokov, NPA **943** (2015) 1; Blaizot, Broniowski & Ollitrault, PRC **90** (2014) 034906; ...

Independent hot spot model

Eigenvalues (label: $\mathrm{IHSM}^{\sigma_{\rm src}}_{N_{\rm src}}$)



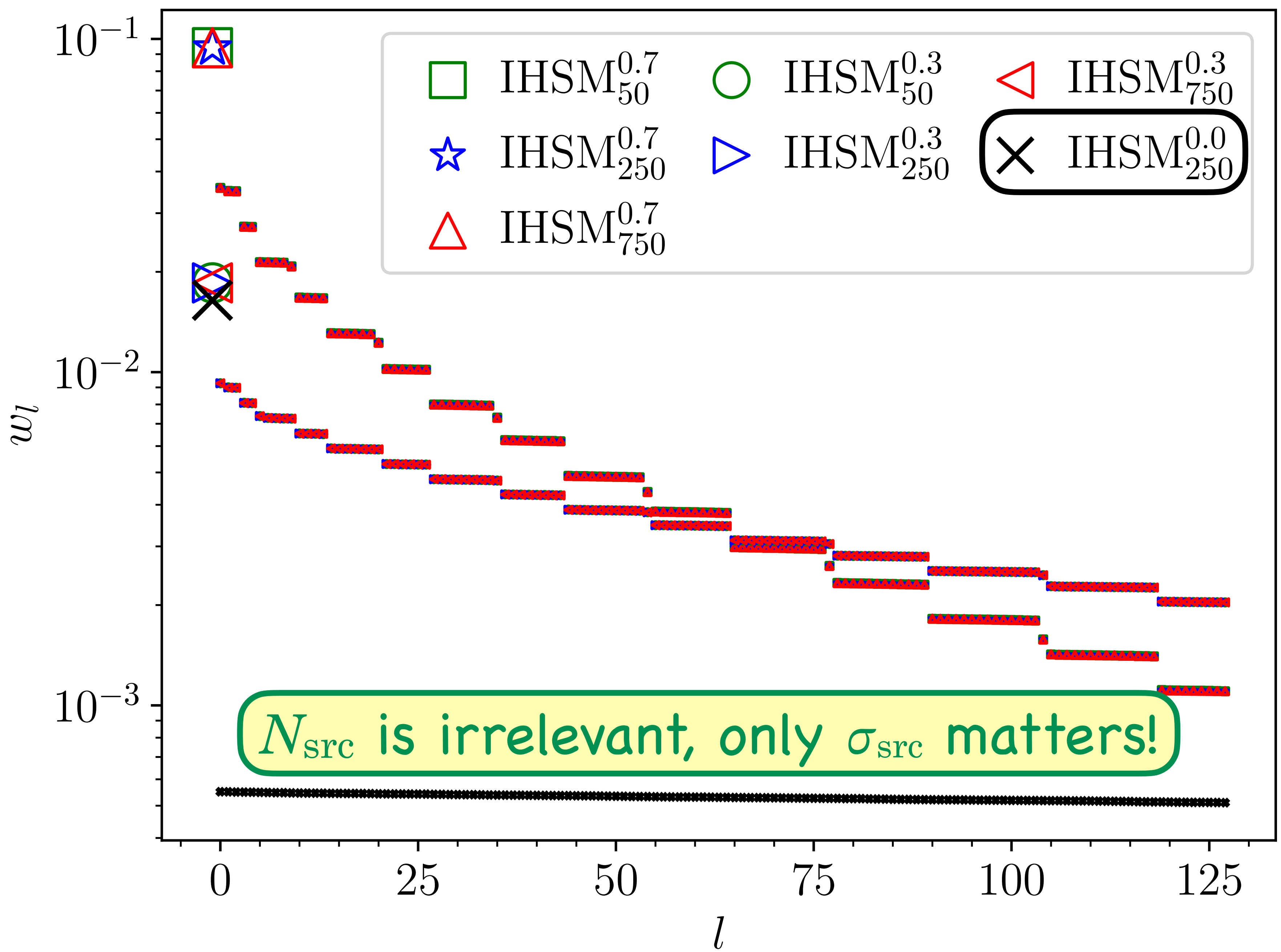
pointlike sources

(almost) degenerate modes!

N.B., H.Roch, A.Schütte, arXiv:2402.07888

Independent hot spot model

Relative weights (label: $\mathrm{IHSM}^{\sigma_{\rm src}}_{N_{\rm src}}$)



pointlike sources

(almost) degenerate modes!

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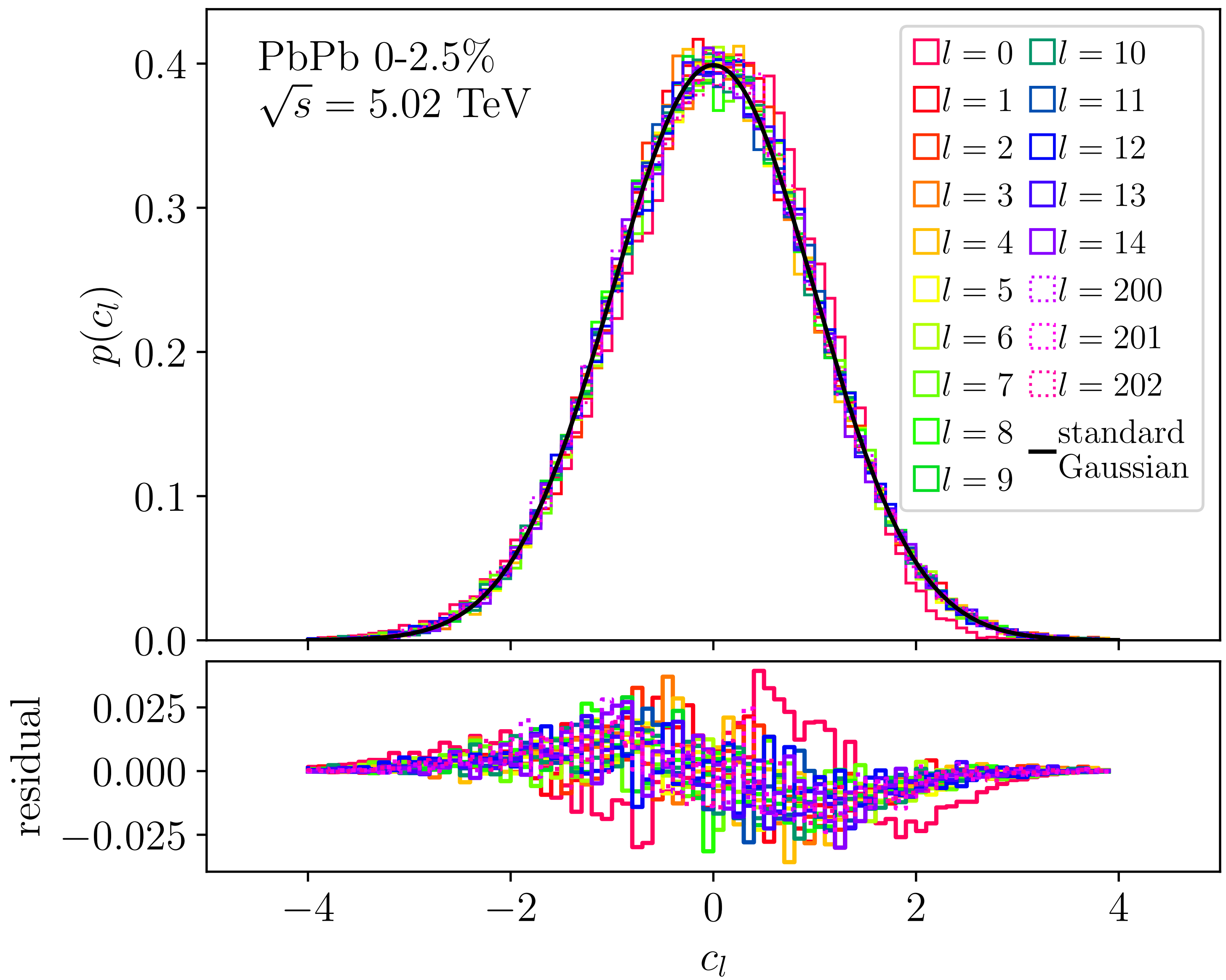
Next step: turn on (controlled) correlations between hot spots...

Statistics of the expansion coefficients

... for a sample of 2^{15} Pb-Pb events



Almost Gaussian statistics, with unit variance (by construction!).

R.Krupczak, N.B., H.Roch, in prep.

Statistics of the expansion coefficients

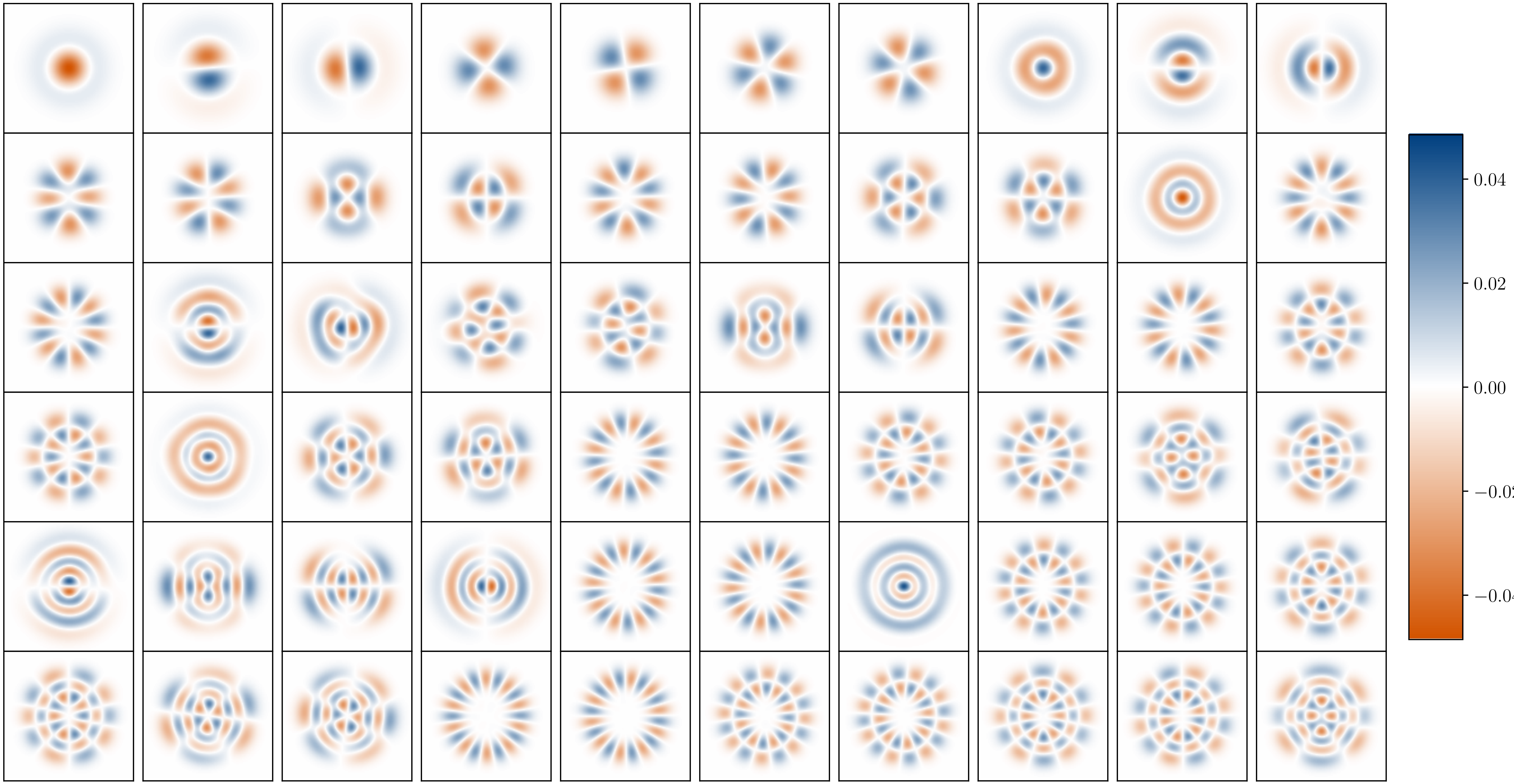
- One can also check (backup slide) that the covariances $\langle c_l c_m \rangle$ are small for $l \neq m$ ☞ the fluctuation modes are uncorrelated.
- But some of the 3-point correlations $\langle c_k c_l c_m \rangle$ are non-zero (not yet systematically investigated; nor higher orders).

  ☞ the fluctuation modes are not statistically independent.

Back to the eigenvectors (normalized)

Pb–Pb at 5.02 TeV, 0–2.5% centrality

(MCGlauber, fixed impact-parameter direction)

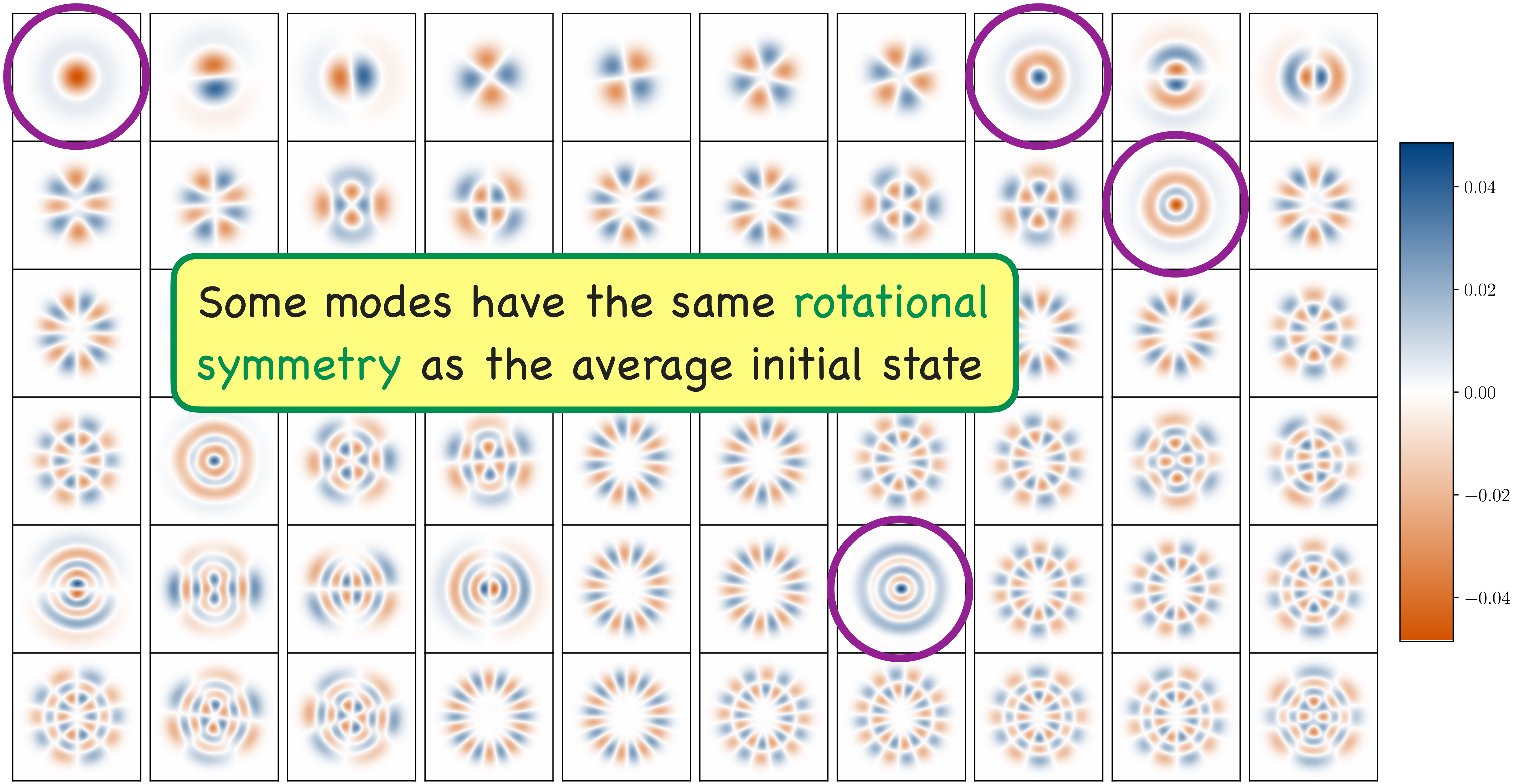


R.Krupczak, N.B., H.Roch, in prep.

Eigenvectors (normalized)

Pb-Pb at 5.02 TeV, 0-2.5% centrality

(nucleon-based MCGlauber, fixed impact-parameter direction)



Some modes have the same rotational symmetry as the average initial state

R.Krupczak, N.B., H.Roch, in prep.

Eigenvectors (normalized)

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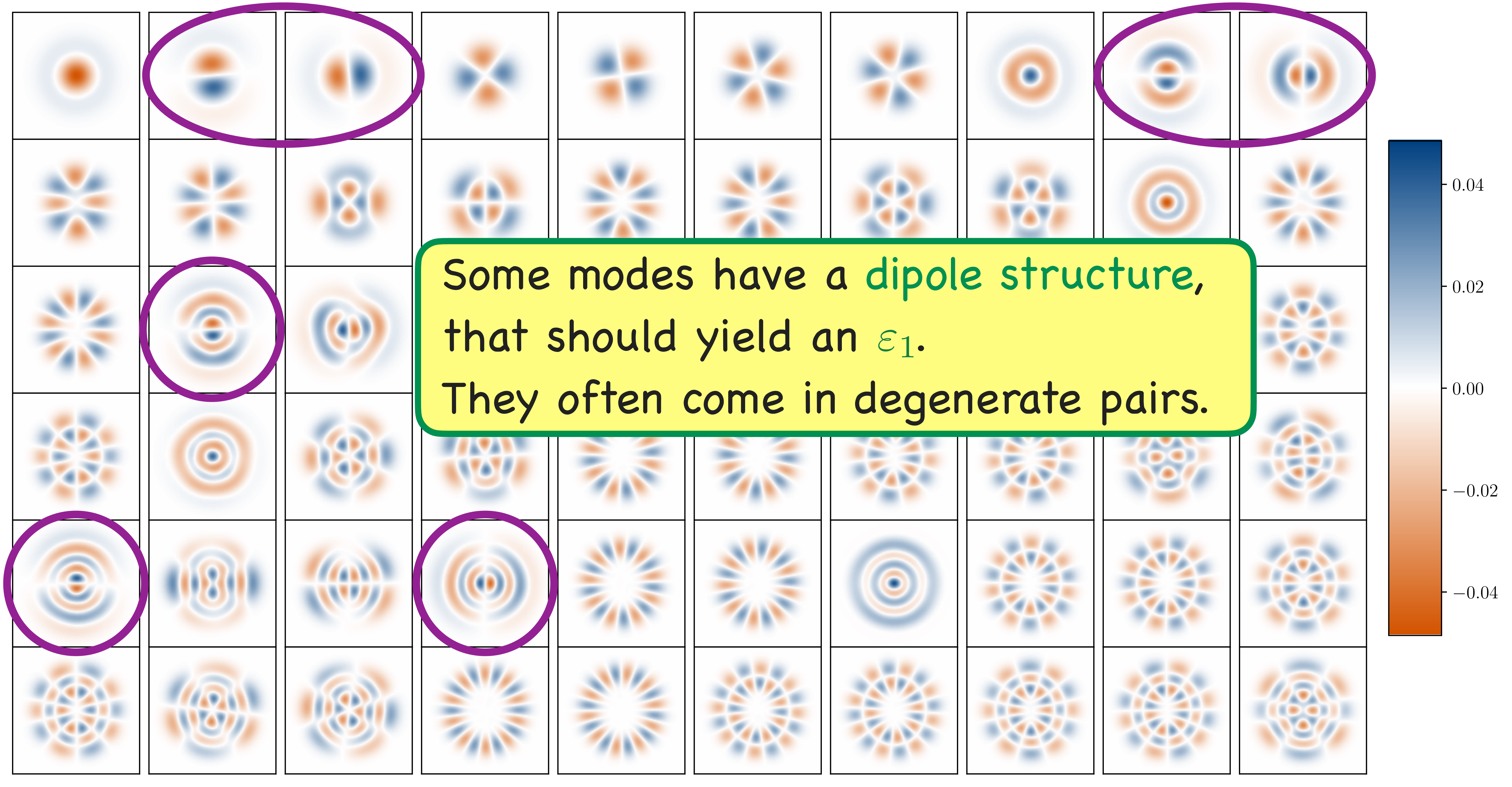

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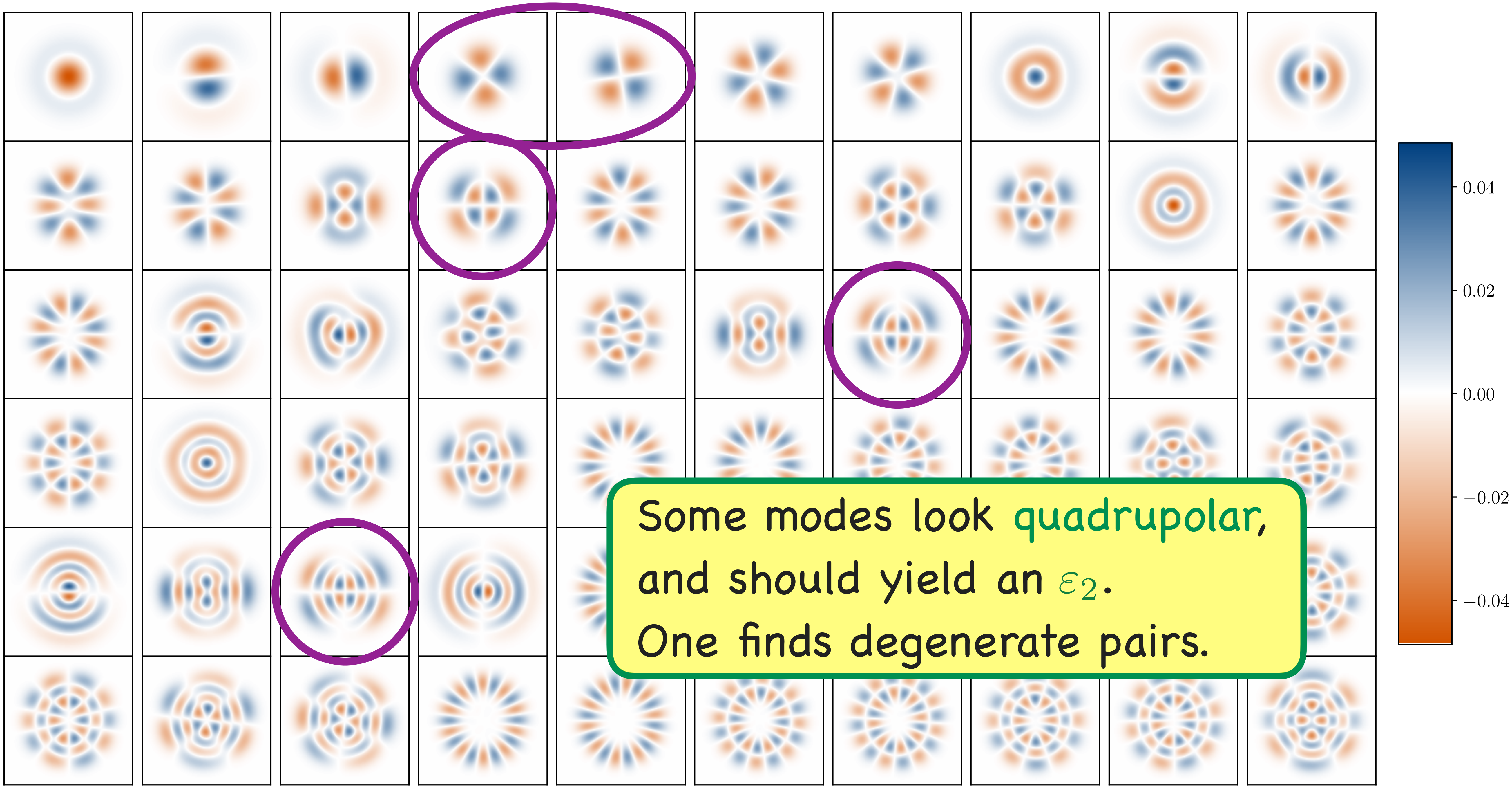

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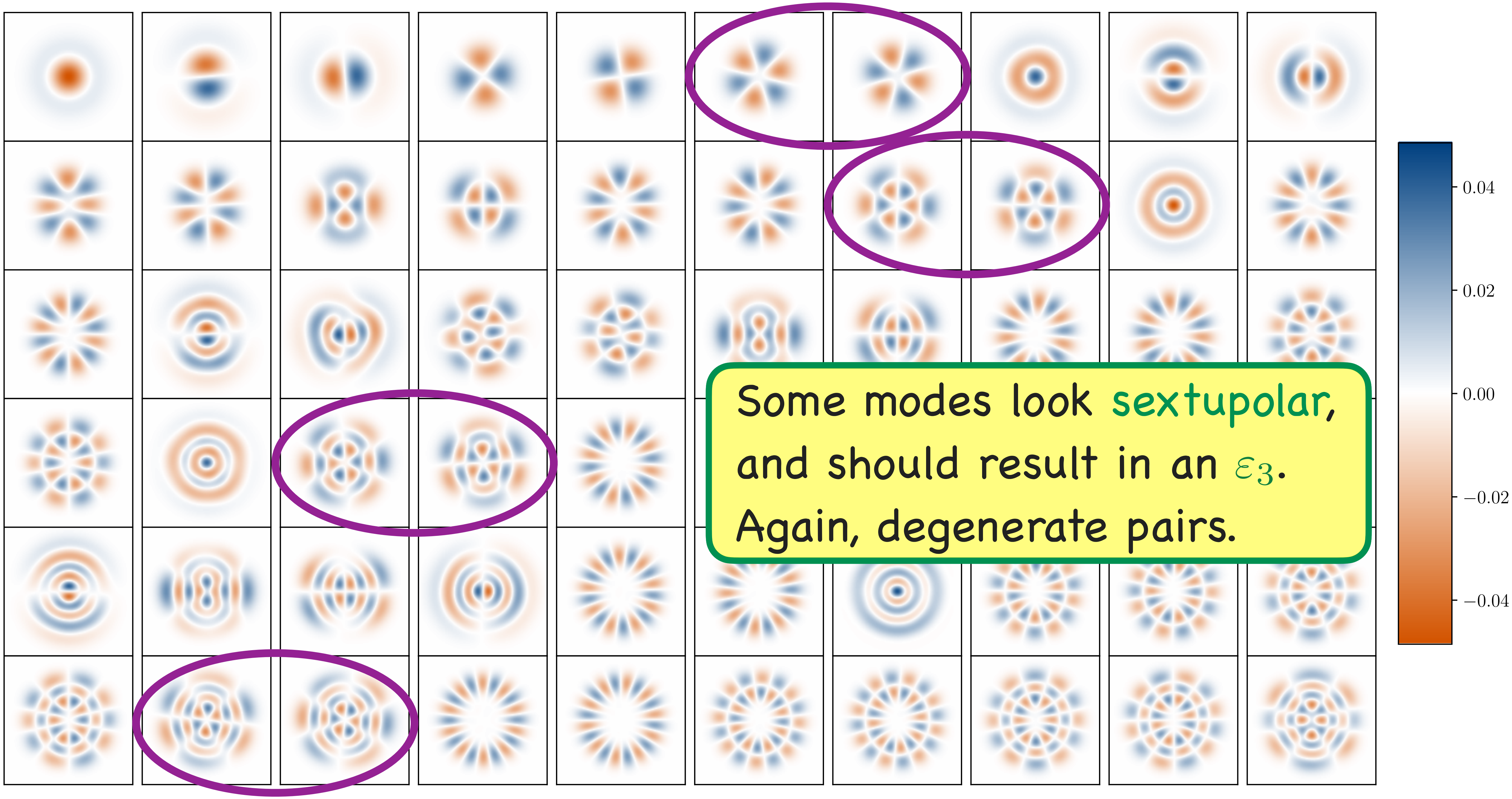

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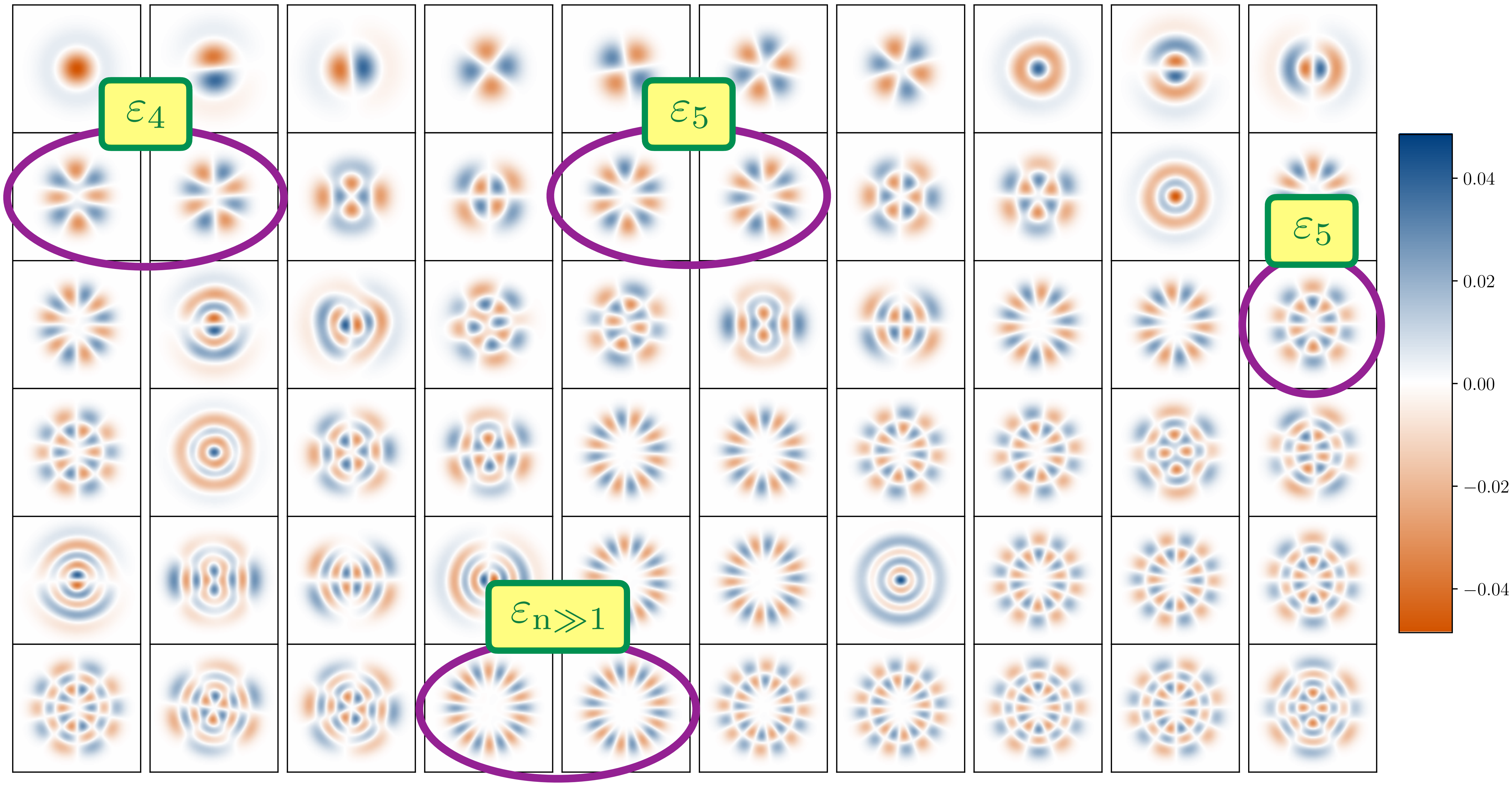

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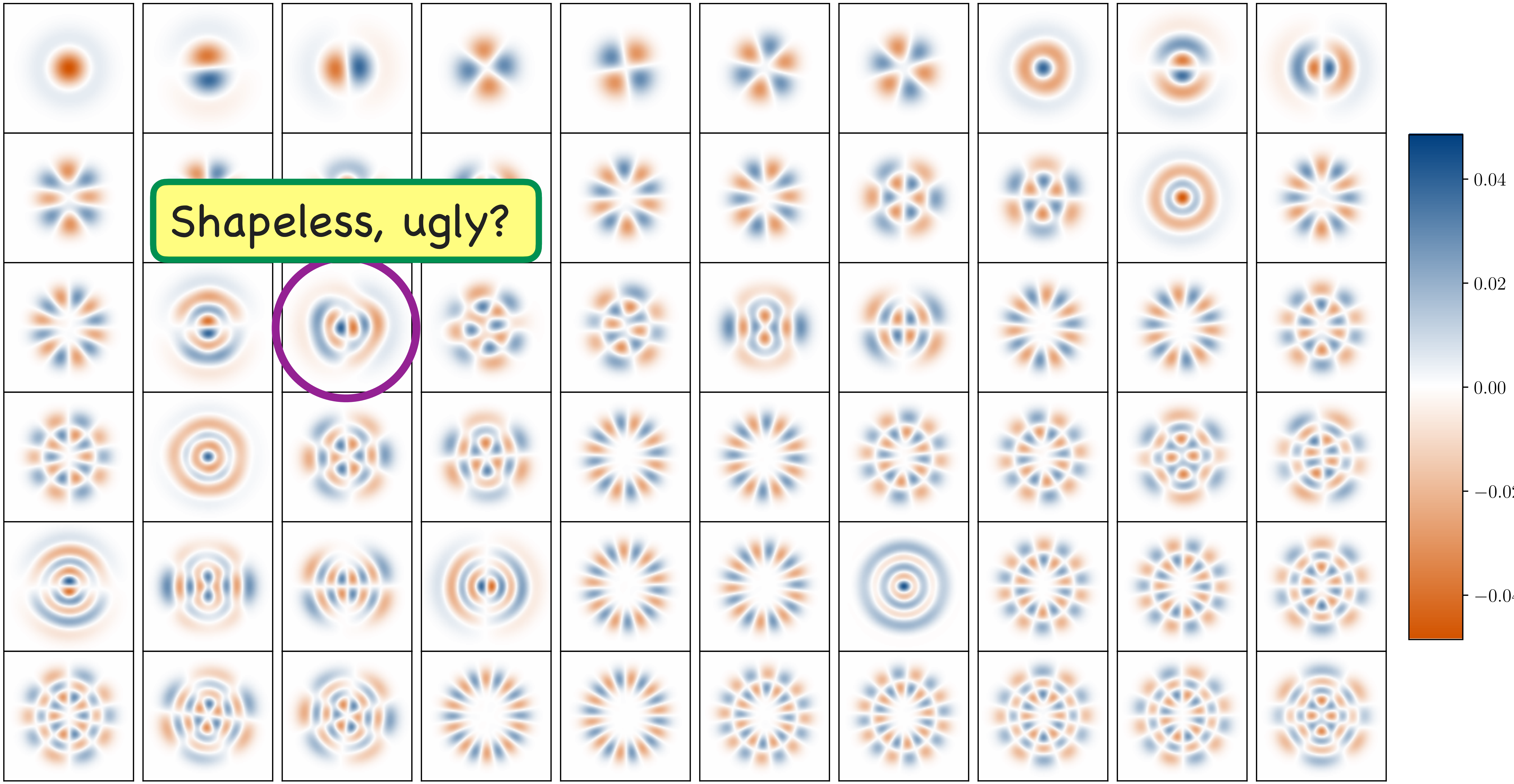

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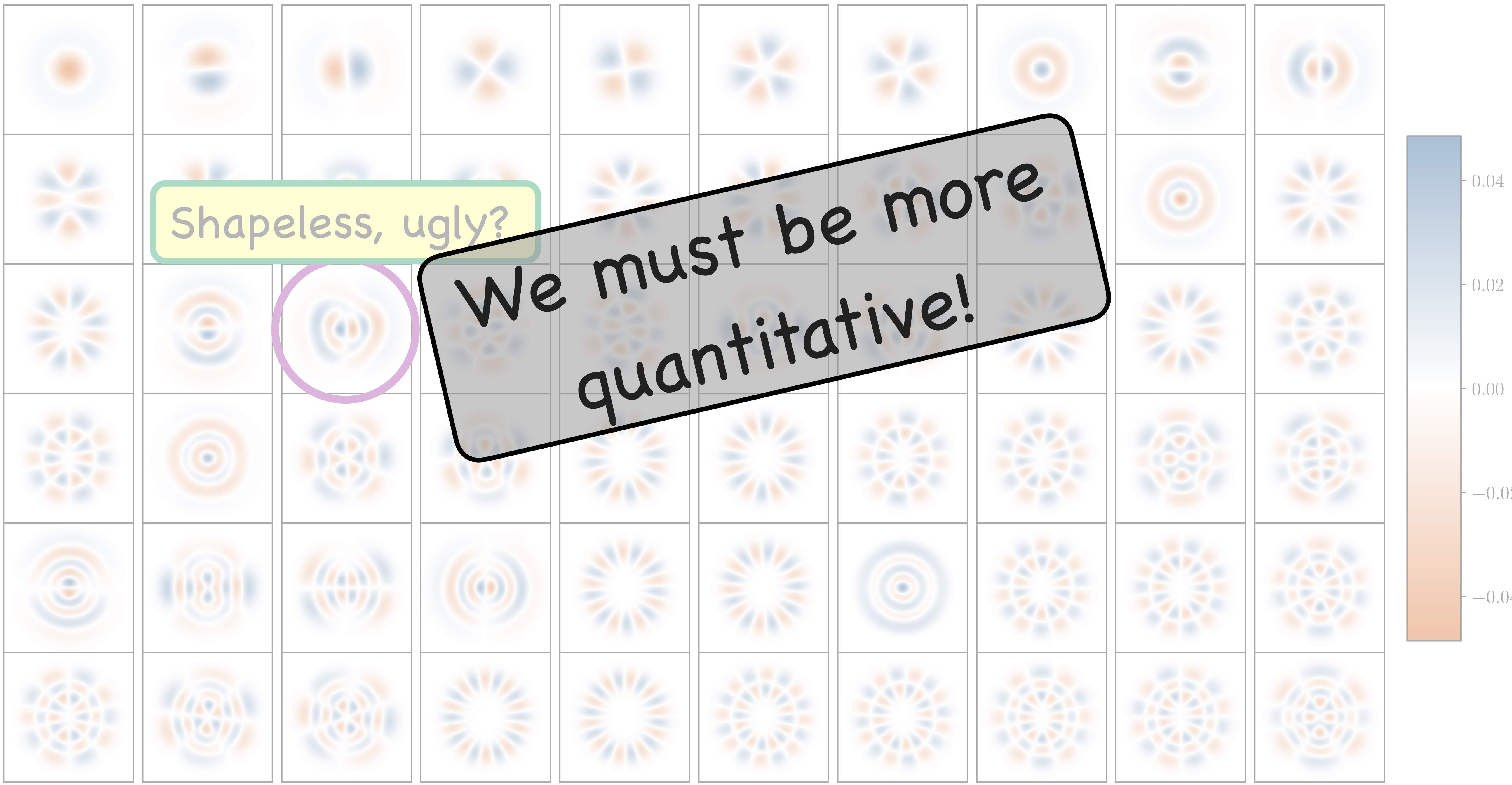


R.Krupczak, N.B., H.Roch, in prep.

Influence of individual fluctuation modes on "observables"

Consider an arbitrary **observable** (from the theorist's point of view*) O_α.

- One can compute it for the average initial state $\bar{\Psi}(\mathbf{x})$:

$$O_\alpha(\bar{\Psi}) \equiv \bar{O}_\alpha$$

- and for the initial state $\Psi_l \equiv \bar{\Psi} + \xi\Psi_l$ for some (small) number ξ:

$$O_\alpha(\bar{\Psi} + \xi\Psi_l) \equiv O^+_{\alpha,l}$$

- If ξ is small enough, one has

$$O_\alpha(\bar{\Psi} + \xi\Psi_l) = O_\alpha(\bar{\Psi}) + L_{\alpha,l}\xi + \frac{Q_{\alpha,ll}}{2}\xi^2 + \mathcal{O}(\xi^3)$$

with $L_{\alpha,l}$ and $Q_{\alpha,ll}$ appropriate (partial) derivatives.

* Final-state observables and initial- & final-state "computables"...

Influence of individual fluctuation modes on initial-state "observables"

Which initial-state observables?

- Eccentricities (using polar coordinates r, θ in the transverse plane):

$$\epsilon_{n,c}(\Phi) \equiv \frac{\displaystyle\int r^n \cos(n\theta)\,\Phi(r,\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta}{\displaystyle\int r^n\, \Phi(r,\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta}\,,\ \epsilon_{n,s}(\Phi) \equiv \frac{\displaystyle\int r^n \sin(n\theta)\,\Phi(r,\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta}{\displaystyle\int r^n\, \Phi(r,\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta}$$

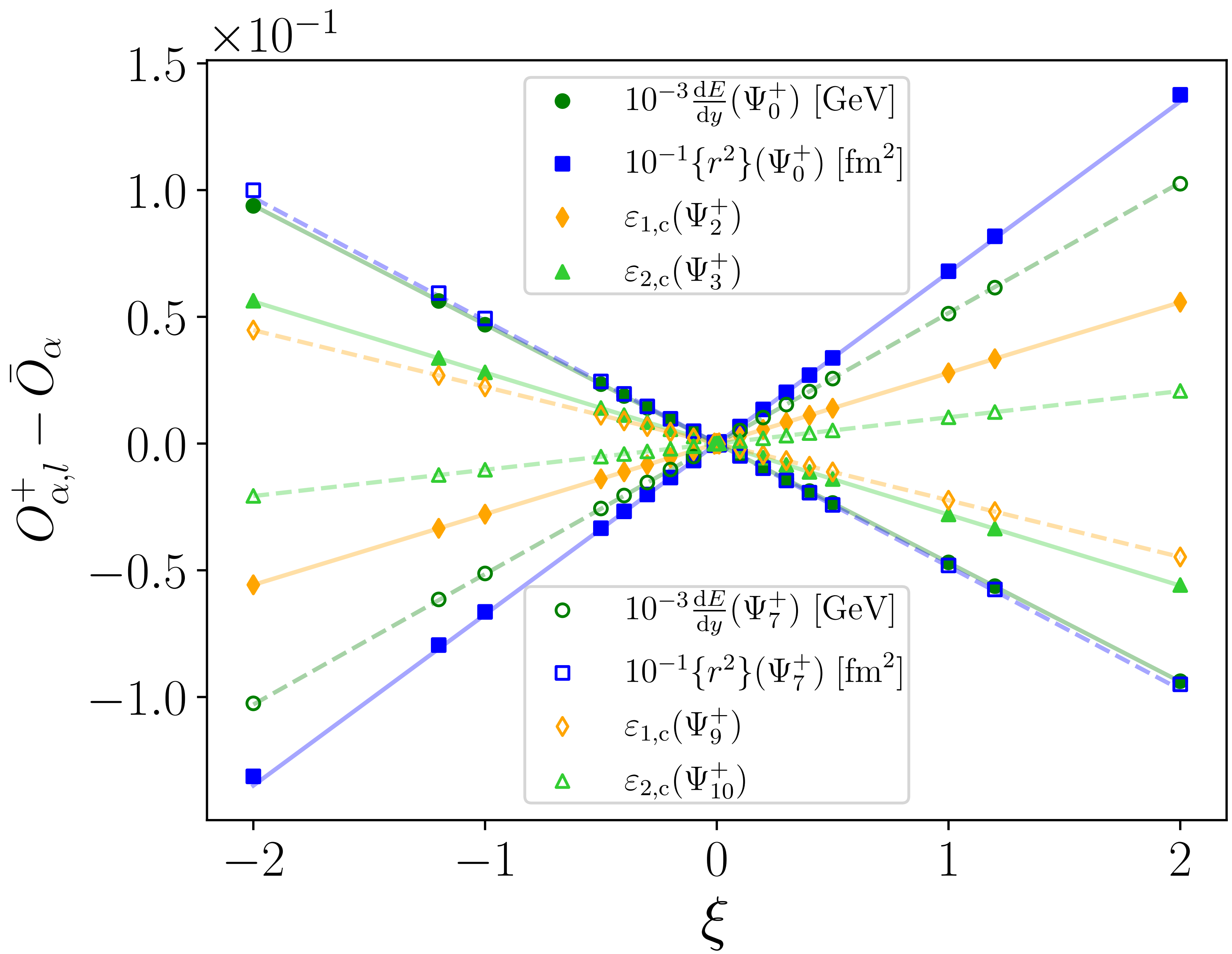
  for $n \geq 2$ (special definition with factor r^3 for $n = 1$);

- total energy (per unit rapidity);

- mean square radius.

Influence of individual fluctuation modes on initial-state "observables"

Pb-Pb at 5.02 TeV, 0-2.5% centrality

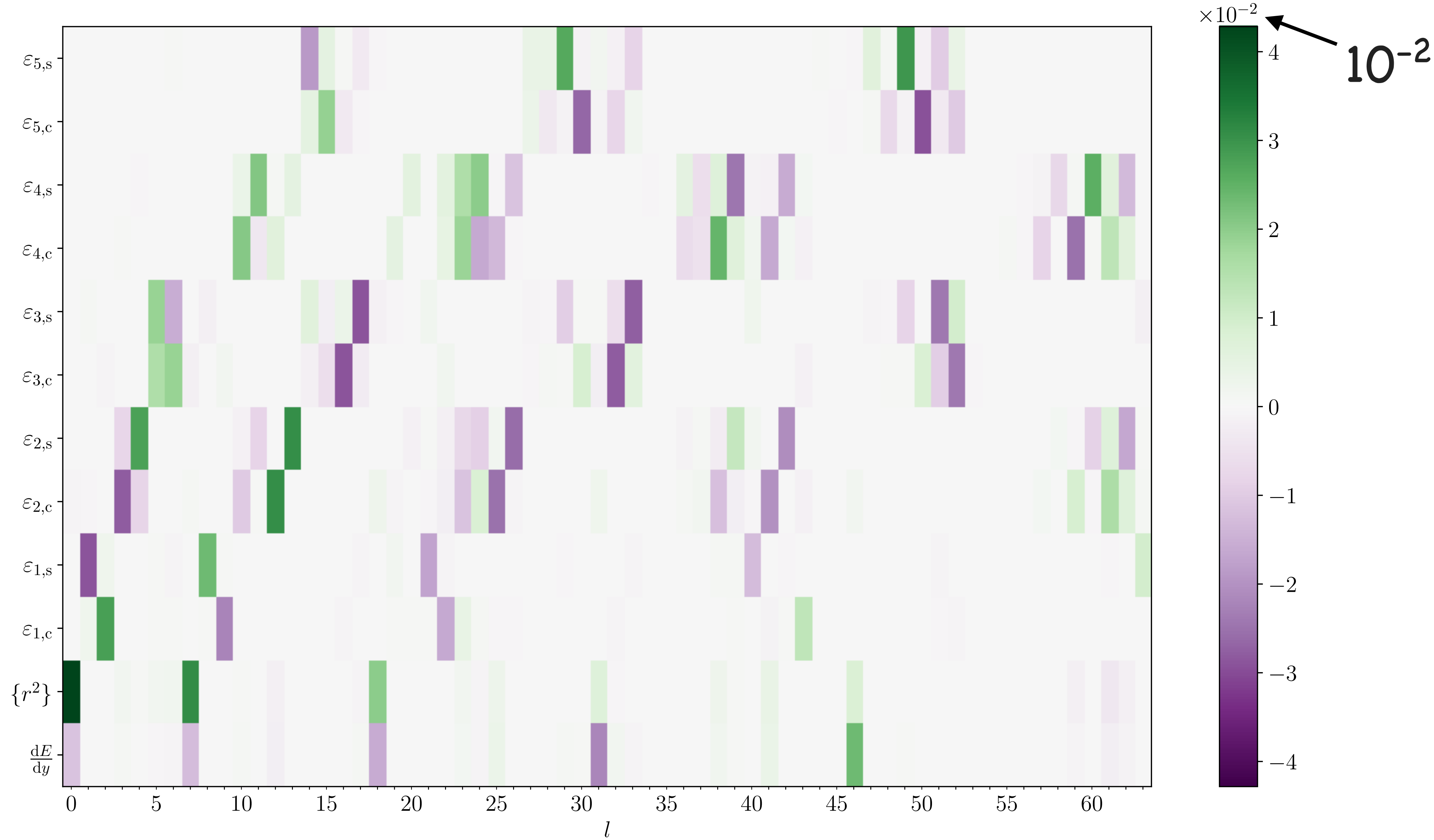


Pretty linear!

R.Krupczak, N.B., H.Roch, in prep.

Influence of individual fluctuation modes on initial-state "observables"

Pb-Pb at 5.02 TeV, 0-2.5% centrality; linear-response coefficients $L_{\alpha,l}$



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N.B., M.Borrell, N.Feld, H.Roch, S.Schlichting, C.Werthmann, PRC **107** (2023) 034905

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Dynamical evolution

Since we have initial states (2D energy densities at τ_0 = 0.2 fm/c ☞ boost invariant system), we can let them evolve dynamically:

- first with KøMPøST: pre-equilibrium stage until $\tau_{\rm hydro}$ = 1 fm/c;
- then with MUSIC, down to $T_{\rm f.o.}$ = 151 MeV;
- then we particlize:
  - either with MUSIC's Cooper–Frye implementation, including the subsequent decays of the free-streaming hadrons;
  - or with iSS, after which we feed the hadrons into SMASH, where they may scatter.

☞ Two kinds of possible final states: at the end of MUSIC (+ free streaming) or at the end of SMASH.

Dynamical evolution

Since we have initial states (2D energy densities at τ_0 = 0.2 fm/c ☞ boost invariant system), we can let them evolve dynamically:

- first with KøMPøST: pre-equilibrium stage until $\tau_{\rm hydro}$ = 1 fm/c;
- then with MUSIC, down to $T_{\rm f.o.}$ = 151 MeV;
- then we particlize.

In particular, we look at mode-by-mode evolution, and consider initial states of the form $\Psi_l \equiv \bar{\Psi} + \xi\Psi_l$, to study the influence of individual fluctuation modes on final-state observables.

Influence of individual fluctuation modes on final-state "observables"

Which final-state observables? For charged hadrons:

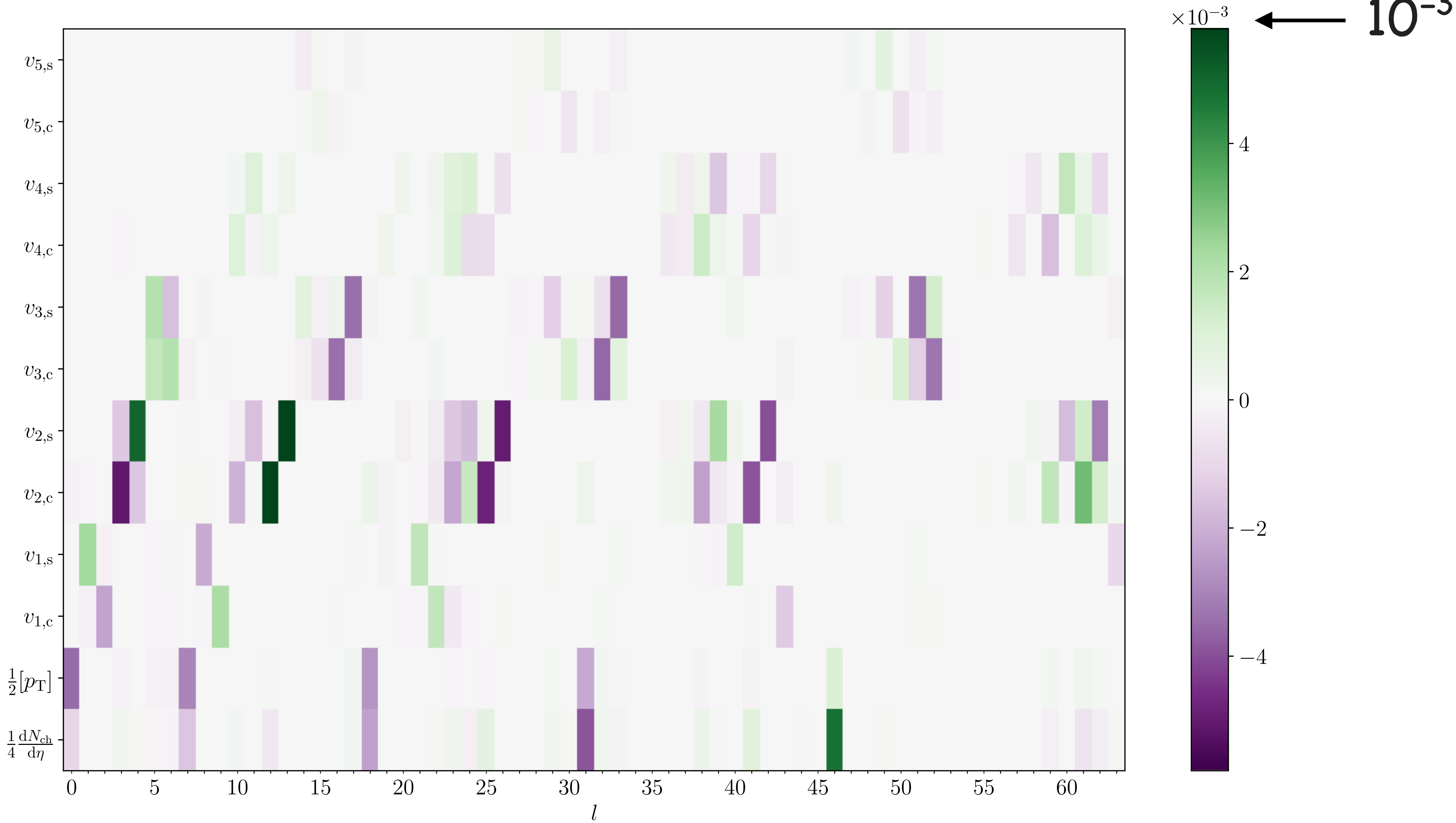
- multiplicity (per unit rapidity);
- average transverse momentum;
- anisotropic flow coefficients: $v_{n,c} \equiv \langle \cos(n\phi_{\mathbf{p}}) \rangle$, $v_{n,s} \equiv \langle \sin(n\phi_{\mathbf{p}}) \rangle$ where $\langle \cdots \rangle$ denotes the average over particles (not events, since we do mode-by-mode evolution).

Reminder from slide 24: the linear and quadratic response coefficients for a given observable O_α are defined such that

$$O_\alpha(\bar{\Psi} + \xi \Psi_l) = O_\alpha(\bar{\Psi}) + L_{\alpha,l}\xi + \frac{Q_{\alpha,ll}}{2}\xi^2 + \mathcal{O}(\xi^3)$$

Influence of individual fluctuation modes on final-state "observables"

Pb-Pb at 5.02 TeV, 0-2.5% centrality; linear-response coefficients $L_{\alpha,l}$ at the end of MUSIC (+ decays).



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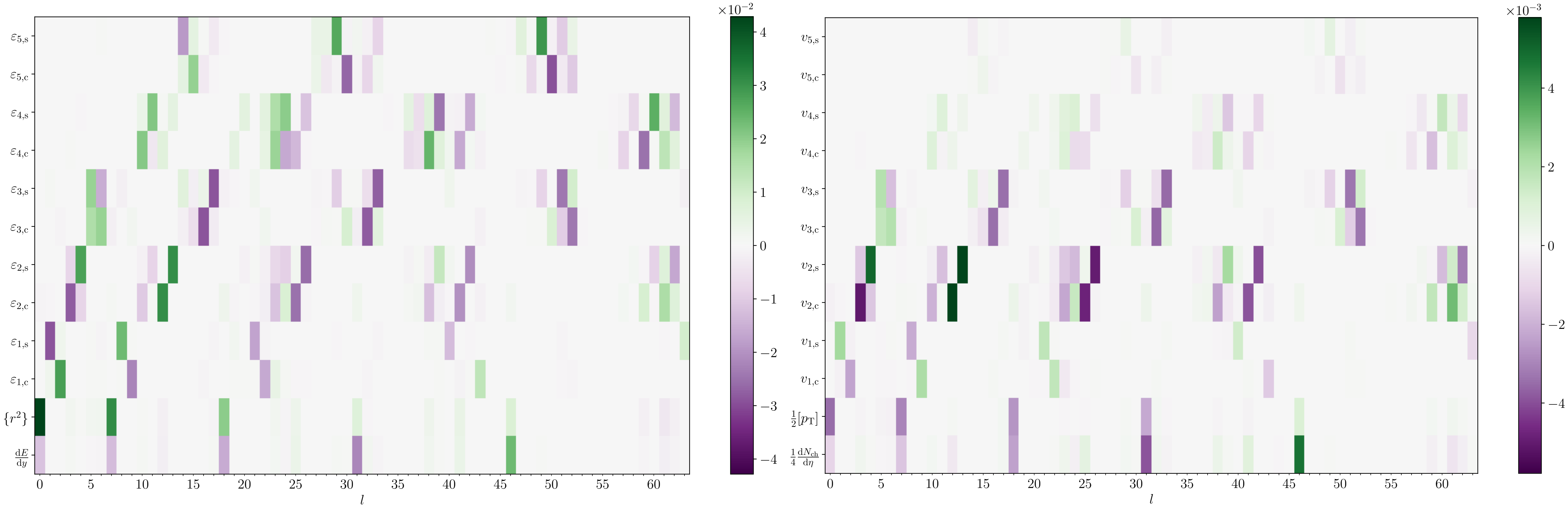
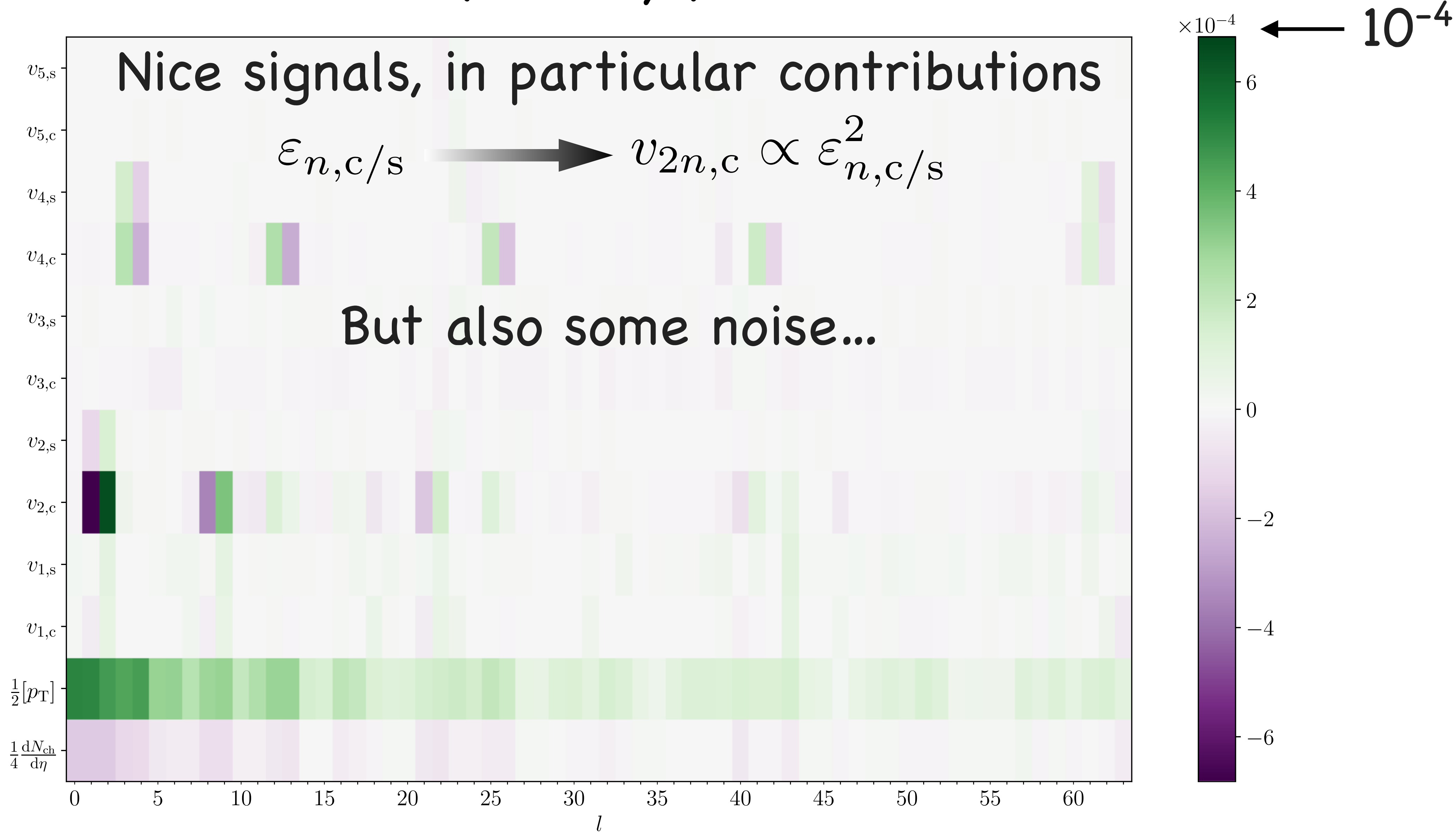
Initial state → Final state



Nice correspondence, e.g. $\varepsilon_{n,c/s} \longrightarrow v_{n,c/s} \propto \varepsilon_{n,c/s}$

Influence of individual fluctuation modes on final-state "observables"

Pb-Pb at 5.02 TeV, 0-2.5% centrality; quadratic-response coefficients $Q_{\alpha,ll}$ at the end of MUSIC (+ decays).



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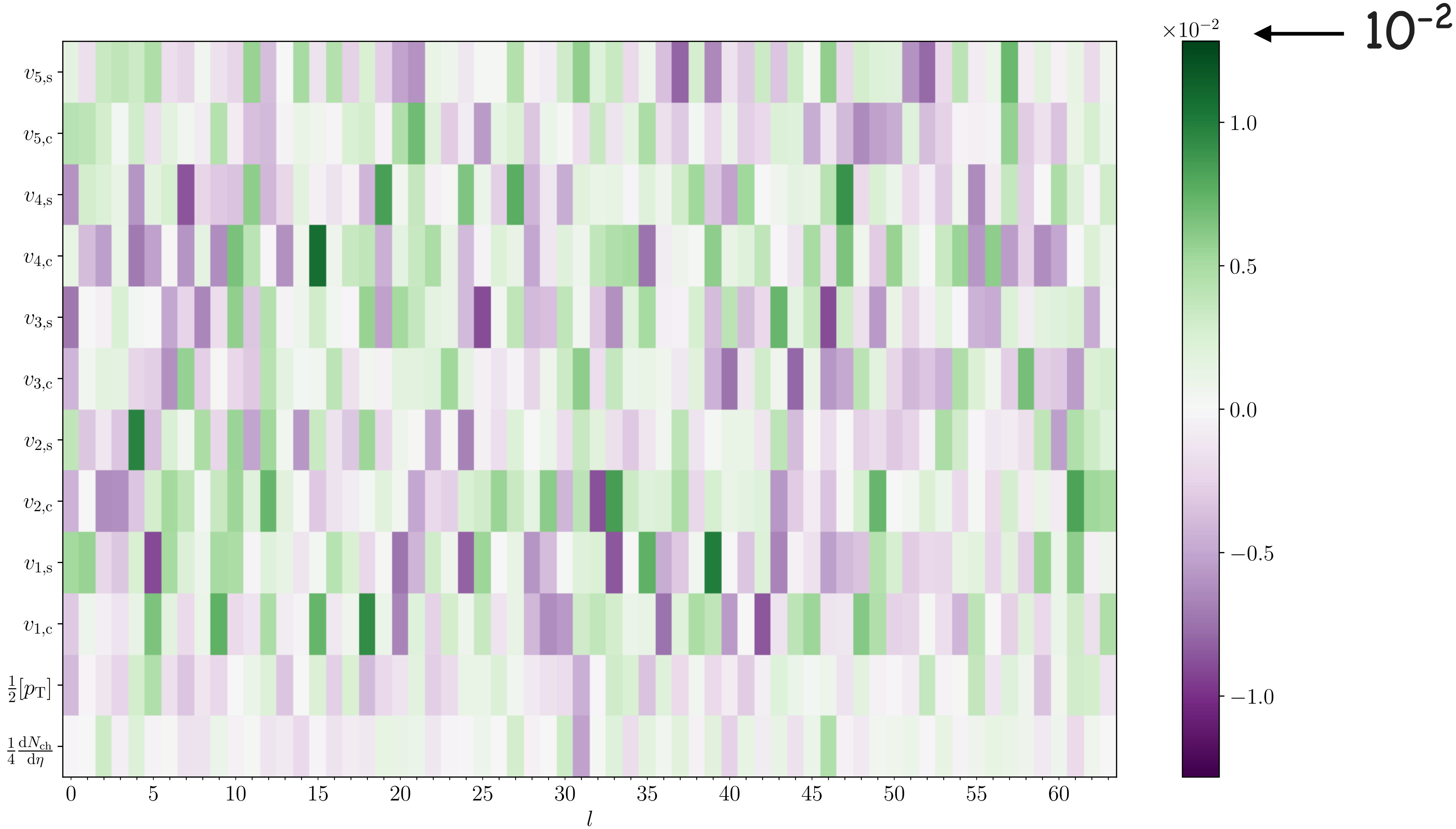
Pb-Pb at 5.02 TeV, 0-2.5% centrality; linear-response coefficients $L_{\alpha,l}$ at the end of MUSIC + iSS + SMASH.



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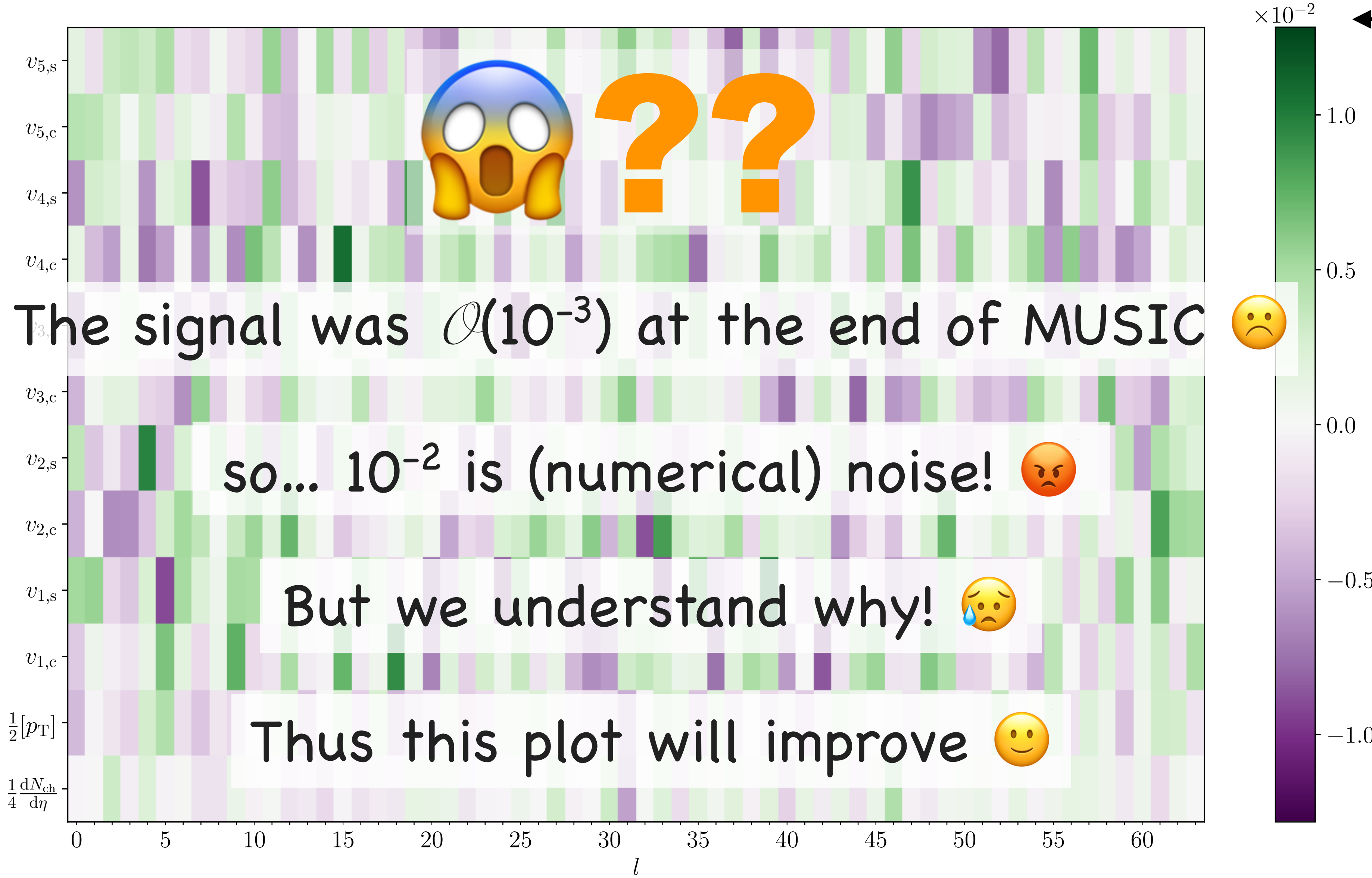


??

10^{-2}

The signal was $\mathcal{O}(10^{-3})$ at the end of MUSIC ☹️

so… 10^{-2} is (numerical) noise! 😡

But we understand why! 😥

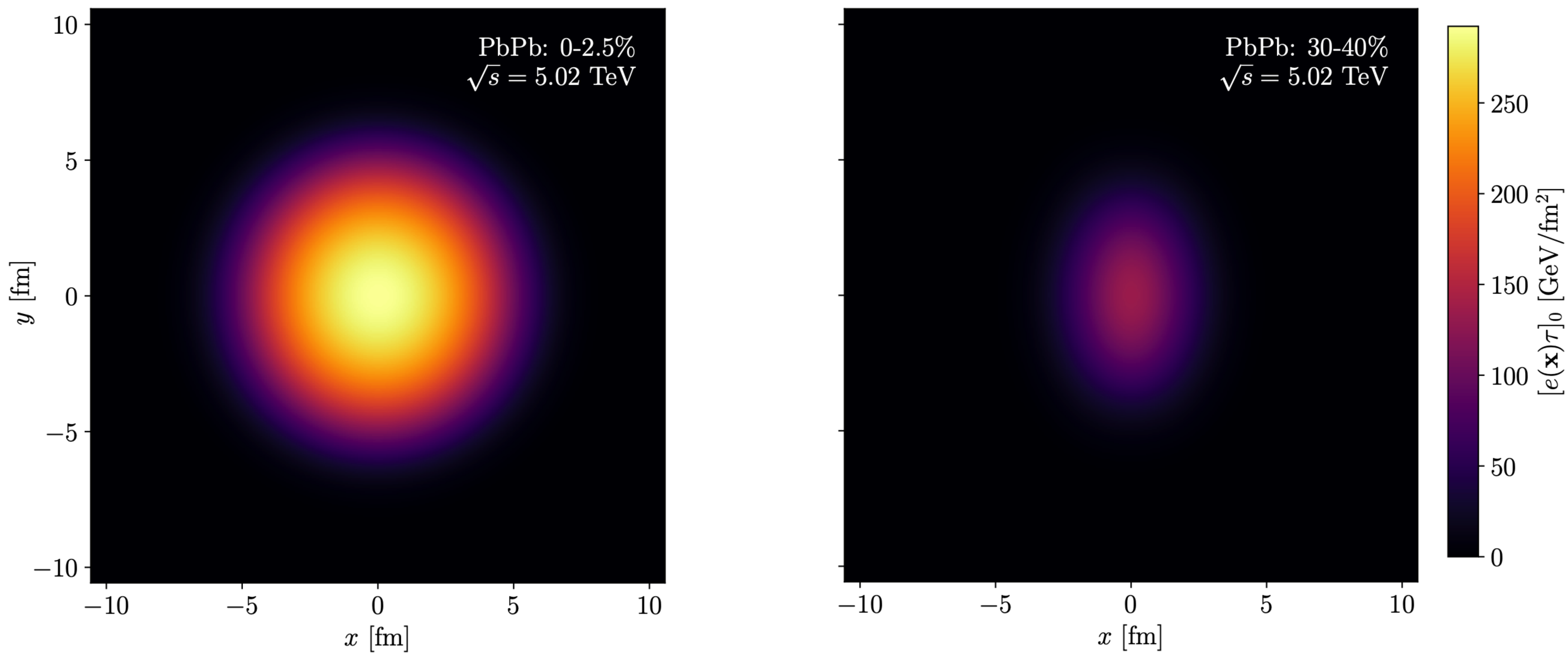
Thus this plot will improve 🙂

~~R.Krupczak, N.B., H.Roch, in prep.~~

Noncentral events: average initial state

Pb-Pb at 5.02 TeV, 30-40% centrality

(MCGlauber, fixed impact-parameter direction)



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Fluctuation modes (normalized)

Pb–Pb at 5.02 TeV, 30–40% centrality

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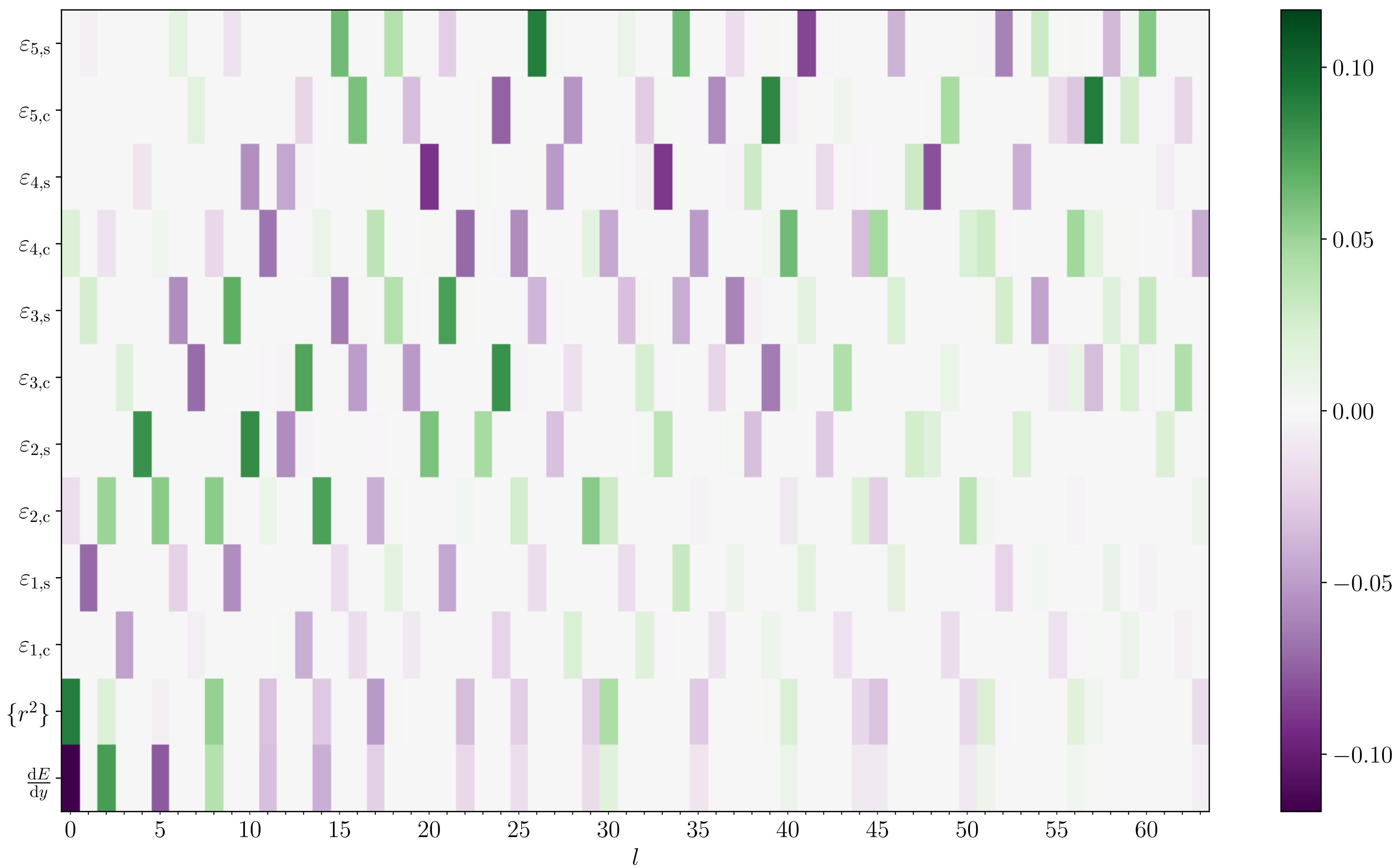

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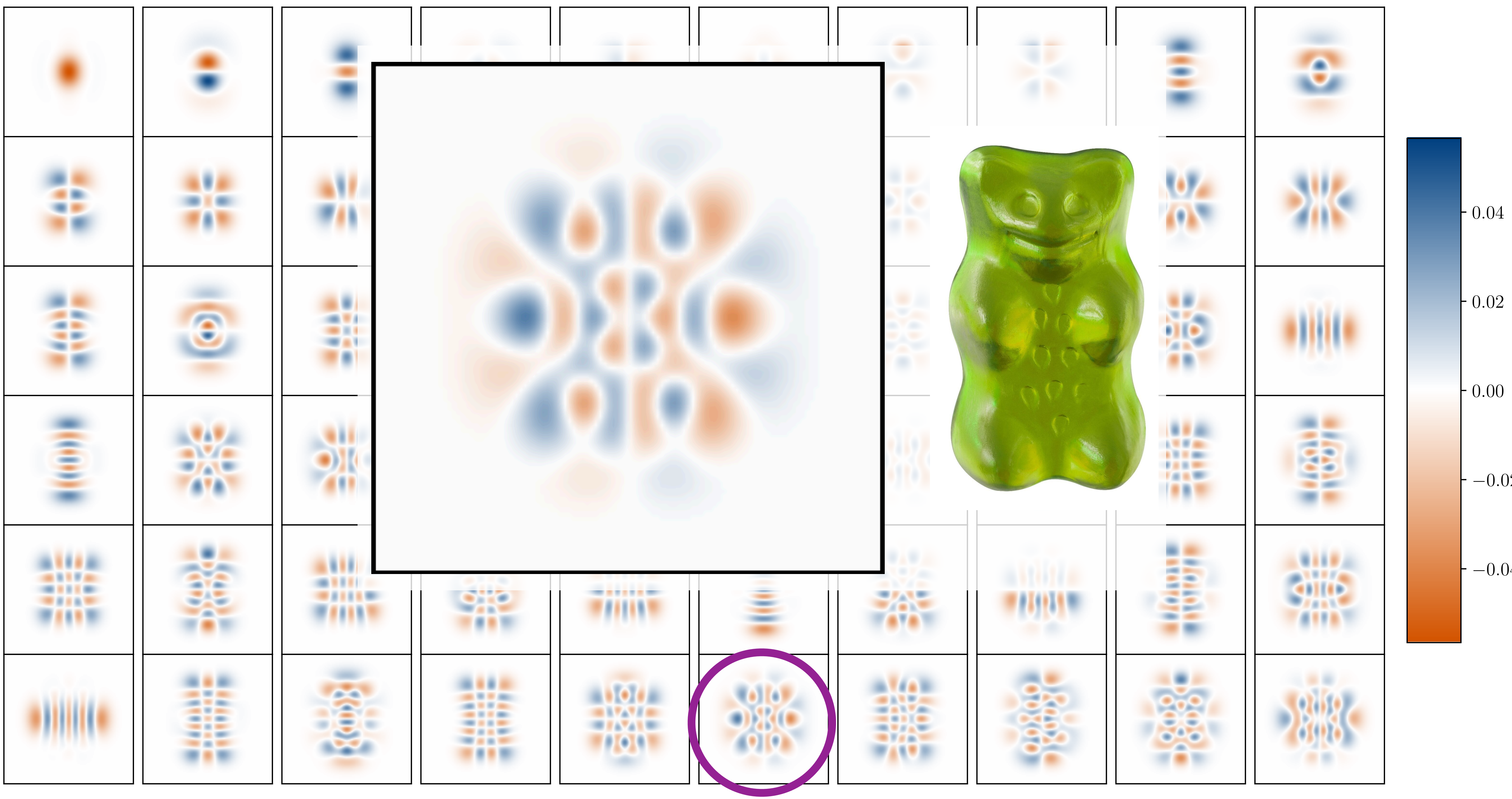
Pb-Pb at 5.02 TeV, 30-40% centrality
(MCGlauber, fixed impact-parameter direction)



R.Krupczak, N.B., H.Roch, in prep.

Statistical description of the initial state fluctuations & mode-by-mode dynamical evolution

- Statistical description of the initial state fluctuations
  - Nice application of linear algebra
  - But computation of the modes can be expansive ⚠️
- Mode-by-mode dynamical evolution
  - Pre-equilibrium + fluid dynamics ☞ works well
  - When adding a hadronic afterburner, extra statistical noise has to be overcome (by the end of this week?).

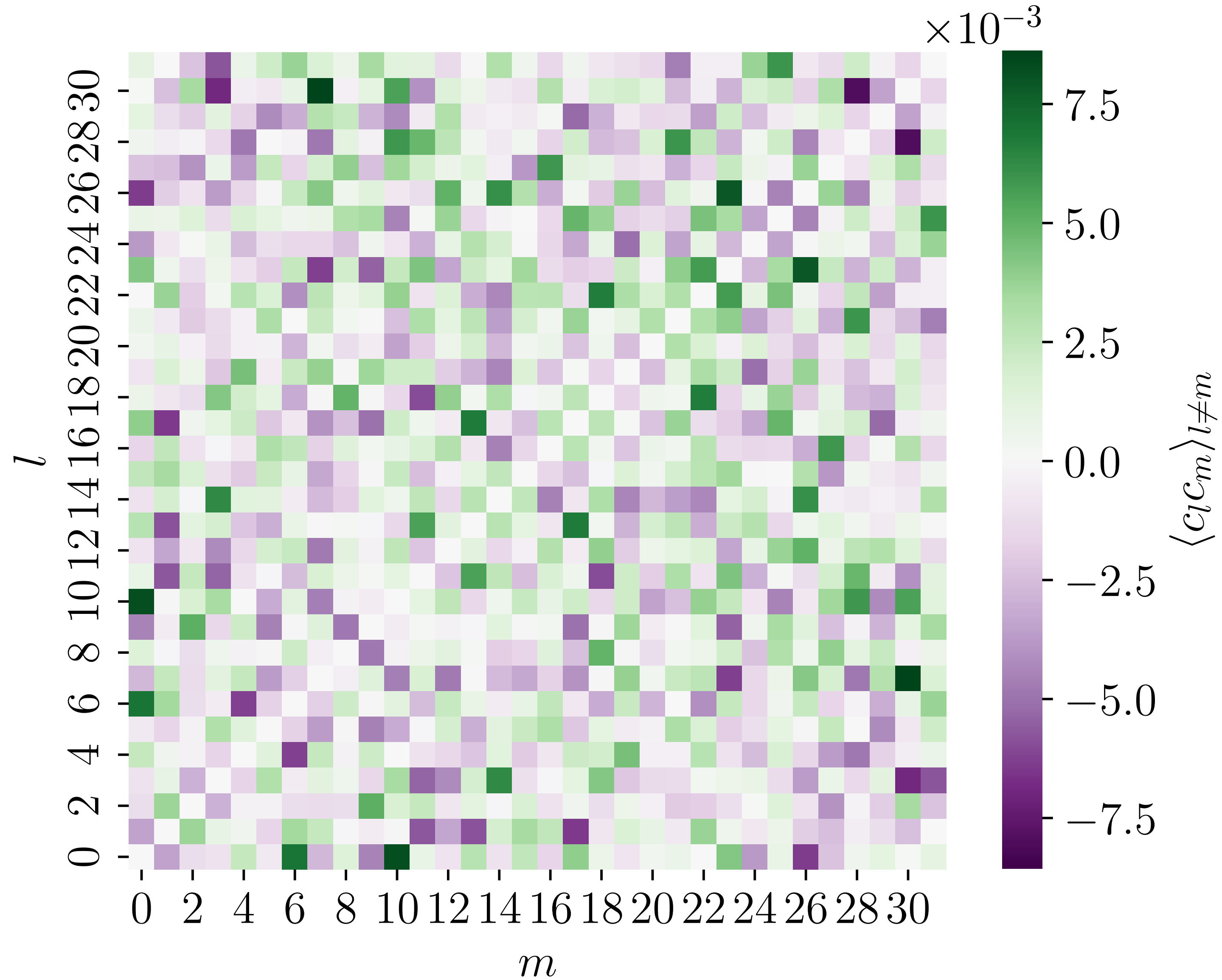
Questions / Ideas / Outlook

- How does the statistical analysis of initial-state fluctuations help?
  - (Dream?) To define a distance between initial-state models?
  - From the response coefficients, one can compute the (co)variances of observables (e.g.: with linear coefs, covariances at order c_l^2)
- Extensions
  - Other systems (e.g.: B.Bachmann, MSc thesis on Ru+Ru vs. Zr+Zr at 200 GeV)
  - More final-state observables (Dream: "golden observables", due to very few modes ☞ reverse engineering)
  - Going 3D; adding conserved charges
  - Inclusion of further effects in toy hot-spot model
  - ...

Extra slides

Statistics of the expansion coefficients

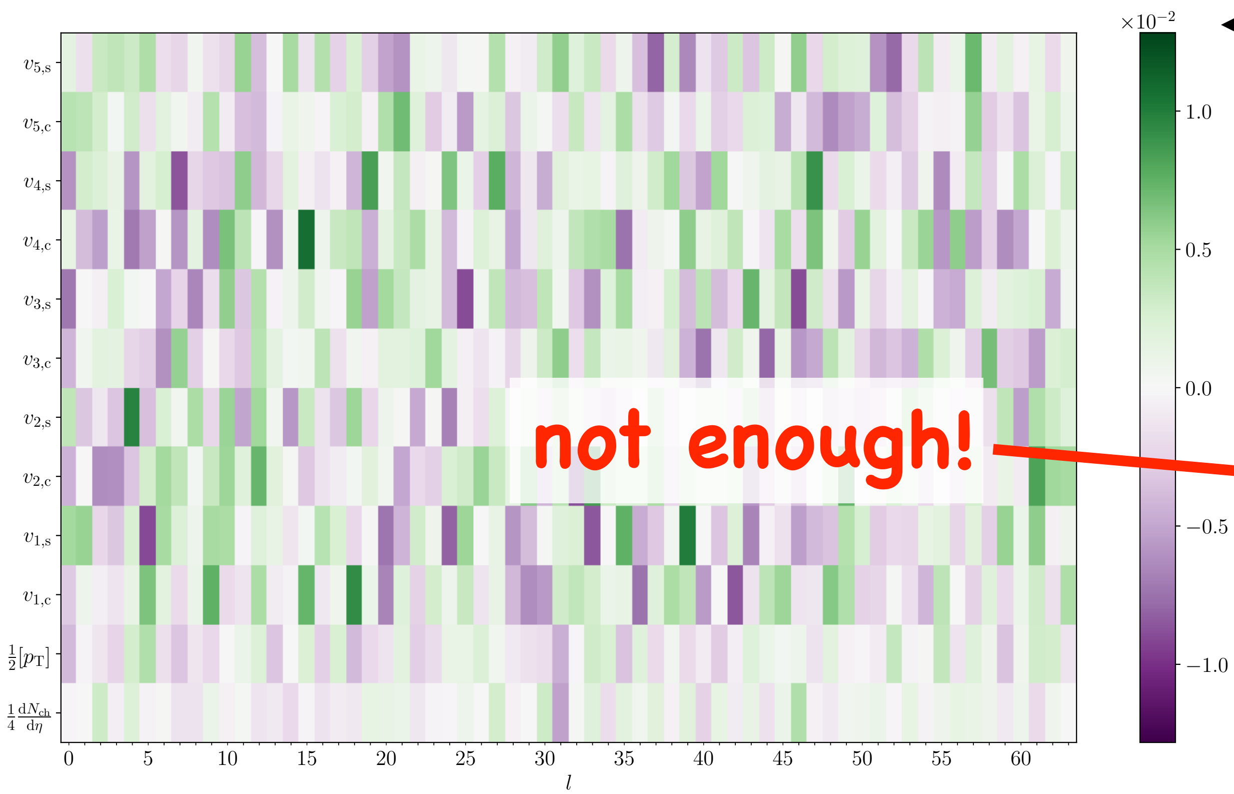
- One can check that the covariances $\langle c_l c_m \rangle$ are small for $l \neq m$.
  - ☞ example in the Independent Hot Spot Model: $\mathrm{IHSM}_{50}^{0.3}$



N.B., H.Roch, A.Schütte, arXiv:2402.07888

Noise in observables in the final state of mode-by-mode dynamics

Linear-response coefficients $L_{\alpha,l}$ at the end of MUSIC + iSS + SMASH.



← 10^{-2}

Do we understand where this order of magnitude comes from? YES!

- Each initial state $\bar{\Psi} + \xi\Psi_l$ leads to a freeze-out hypersurface, from which we produce 1000 SMASH oversamplings, with about 2000 particles (central events).

- All in all, observables are thus computed with $N \approx 2 \times 10^6$ particles
  ☞ numerical noise $\sim 1/\sqrt{N} \approx 10^{-3}$ on every determination of O_α.

- With ξ = 0.1, noise on $L_{\alpha,l} = \dfrac{O_\alpha(\bar{\Psi} + \xi\Psi_l) - O_\alpha(\bar{\Psi} - \xi\Psi_l)}{2\xi}$ is $\approx 10^{-2}$.