Multiparticle correlations due to momentum conservation and statistical jet studies

Nicolas BORGHINI

Universität Bielefeld

Total momentum conservation and statistical studies of jets

- A few useful definitions and properties
  - probability distributions, cumulants, generating functions...

- Multiparticle correlation induced by total momentum conservation
  - a general, model-independent calculation

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- Specific study of two- and three-particle correlations due to total momentum conservation

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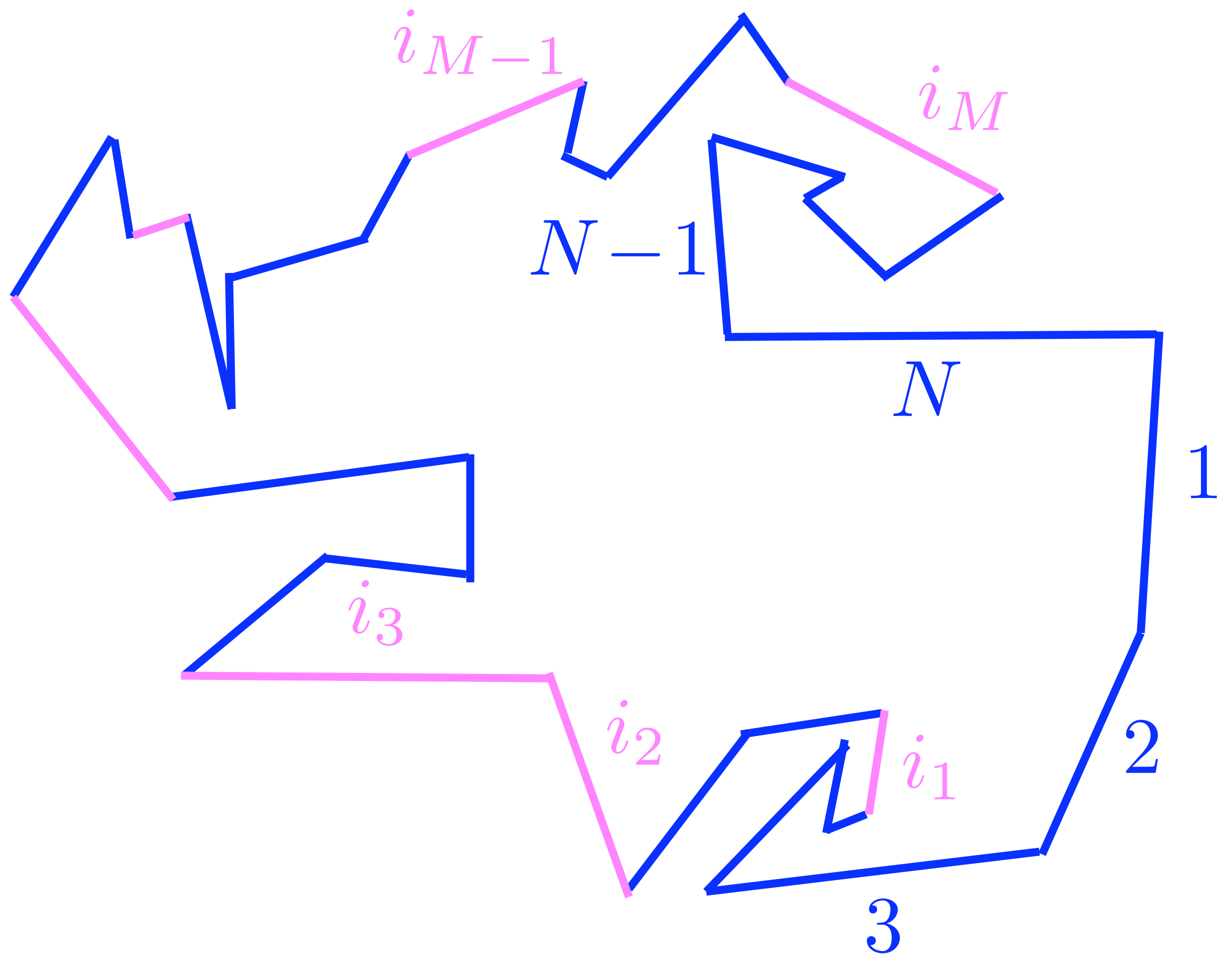
A well-defined mathematical problem...

Consider a finite-size-N ring polymer (in a D-dimensional space):

$\longrightarrow$ "closed": $\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N = \mathbf{0}$



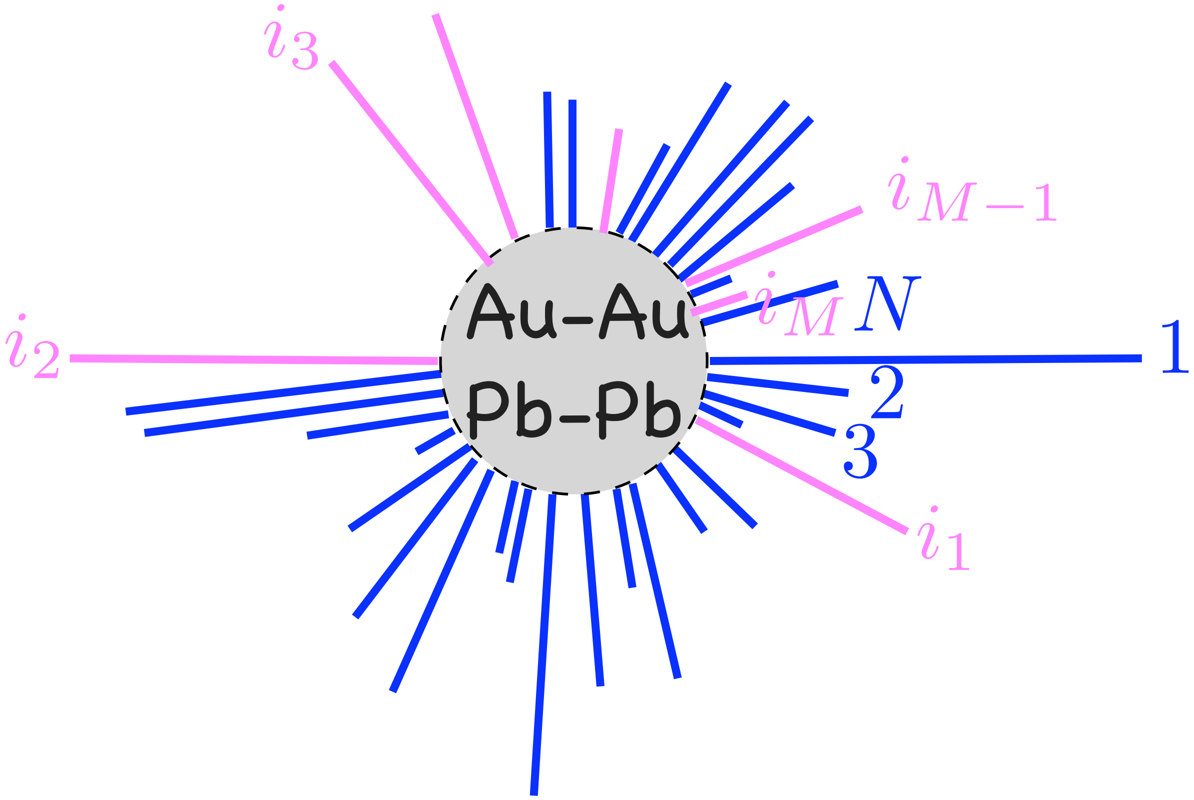
Take M monomers among the N ones.

What is the multiple correlation induced between these monomers by the overall constraint $\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N = \mathbf{0}$?

A well-defined mathematical problem...

Consider N particles constrained by (total) momentum conservation:

for instance, in the center-of-mass frame of the colliding nuclei, the N particles emitted in a Au-Au collision satisfy $\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N = \mathbf{0}$.



What is the correlation between M arbitrary particles induced by the momentum-conservation constraint?

Multiparticle correlations & cumulants

- M-particle probability distribution $f(\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_M})$:

probability that particles $\{i_1, i_2, \ldots, i_M\}$ have momenta $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \ldots, \mathbf{p}_{i_M}$ irrespective of the momenta of the $N-M$ other particles.

☞ normalized to unity: $f(\{\mathbf{p}_{i_k}\}) = \mathcal{O}(1),\ \forall M$

Generating function of the probability distribution:

$$G(x_1,\ldots,x_N) = 1 + x_1\, f(\mathbf{p}_1) + x_2\, f(\mathbf{p}_2) + \ldots + x_1 x_2\, f(\mathbf{p}_1,\mathbf{p}_2) + \ldots$$

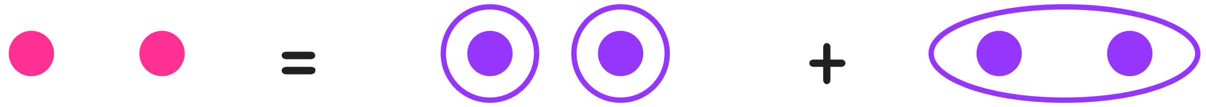
$x_1,\ldots,x_N$ auxiliary (complex) variables

Independent particles: $f(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_N) = f(\mathbf{p}_1)\, f(\mathbf{p}_2)\cdots f(\mathbf{p}_N)$

Multiparticle correlations & cumulants

- M-particle cumulant of the probability distribution $f_c(\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_M})$: connected part of the probability distribution, responsible for the "correlations" (= deviations from statistical independence)

$$f(\mathbf{p}_1,\mathbf{p}_2) = f_c(\mathbf{p}_1)\, f_c(\mathbf{p}_2) + f_c(\mathbf{p}_1,\mathbf{p}_2)$$



(note: $f(\mathbf{p}) = f_c(\mathbf{p})$...)

At the three-particle level:

Generating function of the cumulants:

$$\ln G(x_1,\ldots,x_N) = 1 + x_1\, f_c(\mathbf{p}_1) + x_2\, f_c(\mathbf{p}_2) + \ldots + x_1 x_2\, f_c(\mathbf{p}_1,\mathbf{p}_2) + \ldots$$

Multiparticle correlations & cumulants

How do cumulants scale with the total multiplicity N?

For a system made of independent sub-systems (or with short-range correlations), the probability distributions add up:

$$f(\{\mathbf{p}_j\}) = \sum_A \frac{N_A}{N} f_A(\{\mathbf{p}_j\}) \quad \text{i.e.} \quad G(\{x_j\}) = \prod_A g_A\left(\left\{\frac{N_A\, x_j}{N}\right\}\right)$$

At the cumulant level, $\ln G(\{x_j\}) = \sum_A \ln g_A\left(\left\{\frac{N_A\, x_j}{N}\right\}\right)$

Expand, search for the coefficient of $x_{i_1} \ldots x_{i_M}$

☛ $$f_c(\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_M}) = \mathcal{O}\left(\frac{1}{N^{M-1}}\right)$$

What about the case of particles whose momenta are constrained by total momentum conservation?

Total momentum conservation and M-particle distribution

In the presence of the constraint from $\underbrace{\text{total momentum conservation}}$, the M-particle distribution reads:

single-particle distribution
in the absence of constraint

normalization constant

$$f(\mathbf{p}_1,\ldots,\mathbf{p}_M) \equiv \frac{\left(\prod_{j=1}^{M} F(\mathbf{p}_j)\right)\int \delta^D(\mathbf{p}_1+\cdots+\mathbf{p}_N) \prod_{j=M+1}^{N}\left[F(\mathbf{p}_j)\,\mathrm{d}^D\mathbf{p}_j\right]/\mathcal{N}_D^{N-M}}{\int \delta^D(\mathbf{p}_1+\cdots+\mathbf{p}_N)\prod_{j=1}^{N}\left[F(\mathbf{p}_j)\,\mathrm{d}^D\mathbf{p}_j\right]/\mathcal{N}_D^{N}}$$

M-independent denominator $\equiv 1/\mathcal{C}_D$

which one then inserts in the generating function...

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M-independent denominator $\equiv 1/\mathcal{C}_D$

$$\delta^D(\mathbf{p}_1+\cdots+\mathbf{p}_N) \longrightarrow \int\frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}\prod_{j=1}^{N}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{p}_j}$$

which one then inserts in the generating function...

Generating function

Introducing the notation $\langle g(\mathbf{p})\rangle \equiv \int g(\mathbf{p})F(\mathbf{p})\,\mathrm{d}^D\mathbf{p}/\mathcal{N}_D$, one finds:

$$G(x_1,\ldots,x_N) = \mathcal{C}_D \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}\langle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{p}}\rangle^N \exp\left(\sum_{j=1}^{N} \underbrace{x_j F(\mathbf{p}_j)} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{p}_j}}{\langle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{p}}\rangle}\right)$$

the unmeasurable F is absorbed

$$= \mathcal{C}_D \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D} \exp\left[N \underbrace{\left(\ln\langle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{p}}\rangle + \sum_{j=1}^{N}\frac{\bar{x}_j}{N}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{p}_j}}{\langle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{p}}\rangle}\right)}_{\mathcal{F}(\mathbf{k})}\right]$$

only depends on $\dfrac{x}{N}$

I shall show (using a saddle-point method) that

$$G(x_1,\ldots,x_N) \propto \mathrm{e}^{N\mathcal{F}(\mathbf{k}_0)}\left(1+\sum_{q>l}\frac{x^l}{N^q}\right)$$

Saddle-point method

A Taylor expansion around the saddle-point $\mathbf{k}_0$ yields

$$G(x_1,\dots,x_N) = \mathcal{C}_D\, \mathrm{e}^{N\mathcal{F}(\mathbf{k}_0)} \underbrace{\left(\sum \text{Gaussian integrals}\right)}$$

$$\frac{1}{N^{D/2}}\underbrace{\int\frac{\mathrm{d}^D\kappa}{(2\pi)^D}\,\mathrm{e}^{-\mathcal{F}''(\mathbf{k}_0)\,\kappa^2/2}}_{\frac{1}{[2\pi\underline{\mathcal{F}''(\mathbf{k}_0)}]^{D/2}}}\exp\left[\sum_{m\geq 3}\underline{\frac{\mathcal{F}^{(m)}(\mathbf{k}_0)}{m!}}\,\underbrace{\frac{\kappa^m}{N^{m/2-1}}}_{\leq 1/\sqrt{N}}\right]$$

only depend on $\dfrac{x}{N}$

Therefore $G(x_1,\dots,x_N) = \dfrac{\mathcal{C}_D\,\mathrm{e}^{N\mathcal{F}(\mathbf{k}_0)}}{[2\pi N\mathcal{F}''(\mathbf{k}_0)]^{D/2}}\left(1+\sum_{q>l}\dfrac{x^l}{N^q}\right)$

Cumulants

The generating function of cumulants thus reads

$$\ln G(x_1,\ldots,x_N) = \ln \mathcal{C}_D + \underbrace{N\mathcal{F}(\mathbf{k}_0)}_{\text{function of } \frac{x}{N}} + \ln\left(\text{function of } \frac{x^l}{N^{q\geq l}}\right)$$

independent of x → $\mathcal{C}_D$

function of $\frac{x}{N}$

$\mathcal{F}$ only depends on $\frac{x}{N}$

$\mathbf{k}_0$ function of $\frac{x}{N}$
(solution of $\mathcal{F}'(\mathbf{k}_0) = 0$)

Hence the cumulants:

$$f_c(\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_M}) = \begin{array}{c}\text{coef. of } x_{i_1}\cdots x_{i_M}\\ \text{in } N\mathcal{F}(\mathbf{k}_0)\end{array} + \mathcal{O}\left(\frac{1}{N^M}\right) = \mathcal{O}\left(\frac{1}{N^{M-1}}\right)$$

The cumulants arising from total momentum conservation follow the same scaling behaviour as those from short-range correlations!

☞ nice for "cumulant" or "Lee-Yang zeroes" methods of anisotropic-flow analysis

Computing the first cumulants

- The saddle-point is given by $\boxed{\mathcal{F}'(\mathbf{k}_0) = 0}$, i.e.

$$\left(\sum_{j=1}^{N} \frac{x_j}{N} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{p}_j}}{\langle \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{p}}\rangle} - 1\right) \langle \mathbf{p}\,\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{p}}\rangle = \sum_{j=1}^{N} \frac{x_j}{N} \mathbf{p}_j \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{p}_j}$$

- The cumulants are given by $\boxed{\ln G(x_1, \ldots, x_N) = N\mathcal{F}(\mathbf{k}_0)}$

To lowest order*, $\mathrm{i}\mathbf{k}_0 = -\frac{D}{\langle \mathbf{p}^2\rangle} \sum_{j=1}^{N} \frac{x_j}{N} \mathbf{p}_j$, hence

$$\mathcal{F}(\mathbf{k}_0) = \sum_{j=1}^{N} \frac{x_j}{N} - \frac{D}{2\langle \mathbf{p}^2\rangle} \left(\sum_{j=1}^{N} \frac{x_j}{N} \mathbf{p}_j\right)^2$$

which gives $f_c(\mathbf{p}_1, \mathbf{p}_2) = -\frac{D\,\mathbf{p}_1\cdot\mathbf{p}_2}{N\langle \mathbf{p}^2\rangle}$, of order $\mathcal{O}\left(\frac{1}{N}\right)$ as expected

* assuming $F(\mathbf{p})$ isotropic, so that $\langle \mathbf{p}\rangle = 0$ and $\langle(\mathbf{k}_0\cdot\mathbf{p})^2\rangle = \mathbf{k}_0{}^2\langle \mathbf{p}^2\rangle/D$

Computing the first cumulants

Going to the next order in $\frac{x}{N}$:

$$\mathrm{i}\mathbf{k}_0 = -\left[1_D - \left(X_0 1_D - \frac{D}{\langle \mathbf{p}^2\rangle} X_2\right)\right]^{-1} \frac{D}{\langle \mathbf{p}^2\rangle}\mathbf{X_1}$$

unit $D \times D$ matrix

with $X_0 \equiv \sum_{j=1}^{N} \frac{x_j}{N}$, $\mathbf{X_1} \equiv \sum_{j=1}^{N} \frac{x_j}{N}\mathbf{p}_j$, $X_2 \equiv \sum_{j=1}^{N} \frac{x_j}{N}\mathbf{p}_j \otimes \mathbf{p}_j$

☞ $$\mathcal{F}(\mathbf{k}_0) = X_0 - \frac{D}{2\langle \mathbf{p}^2\rangle}(\mathbf{X_1})^2 - \frac{D}{2\langle \mathbf{p}^2\rangle}\mathbf{X_1}\cdot\left(X_0 1_D - \frac{D}{\langle \mathbf{p}^2\rangle}X_2\right)\cdot\mathbf{X_1}$$

2-particle cumulants

$$\ln G(x_1,\ldots,x_N) = \sum_{j=1}^{N} x_j \boxed{-\frac{D}{2N\langle \mathbf{p}^2\rangle}\sum_{j,k} x_j x_k (\mathbf{p}_j\cdot\mathbf{p}_k)}$$
$$\boxed{-\frac{D}{2N^2\langle \mathbf{p}^2\rangle}\sum_{j,k,l} x_j x_k x_l\left[\mathbf{p}_j\cdot\mathbf{p}_l - \frac{D}{\langle \mathbf{p}^2\rangle}(\mathbf{p}_j\cdot\mathbf{p}_k)(\mathbf{p}_k\cdot\mathbf{p}_l)\right]}$$

3-particle cumulants: $\mathcal{O}(1/N^2)$!

Total momentum conservation and M-particle cumulants

Using a saddle-point method (which implies $N \gg 1$), I have computed in a model-independent way the multiparticle cumulants arising from the constraint $\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N = \mathbf{0}$

will be taken =2 in what follows (transverse momentum conservation)

$$
\begin{aligned}
f_c(\mathbf{p}_1, \mathbf{p}_2) &= -\frac{\textcircled{D}\,\mathbf{p}_1 \cdot \mathbf{p}_2}{N\langle \mathbf{p}^2 \rangle} \\
f_c(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) &= -\frac{D}{N^2\langle \mathbf{p}^2 \rangle}(\mathbf{p}_1 \cdot \mathbf{p}_2 + \mathbf{p}_1 \cdot \mathbf{p}_3 + \mathbf{p}_2 \cdot \mathbf{p}_3) \\
&\quad + \frac{D^2}{N^2\langle \mathbf{p}^2 \rangle^2}\left[(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_3) + (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{p}_2 \cdot \mathbf{p}_3)\right. \\
&\qquad \left. + (\mathbf{p}_1 \cdot \mathbf{p}_3)(\mathbf{p}_2 \cdot \mathbf{p}_3)\right]
\end{aligned}
$$

Moreover, the M-particle cumulant arising from the conservation of total momentum scales with multiplicity as $1/N^{M-1}$, as those from short-range correlations!

Two-particle correlation due to total transverse momentum conservation

We have seen that $f_c(\mathbf{p}_1, \mathbf{p}_2) = -\frac{2\,\mathbf{p}_1 \cdot \mathbf{p}_2}{N\langle \mathbf{p}^2 \rangle}$, which means that the two-particle probability distribution reads

$$f(\mathbf{p}_1, \mathbf{p}_2) = f(\mathbf{p}_1) f(\mathbf{p}_2) \left(1 - \frac{2\, p_1 p_2 \cos(\varphi_2 - \varphi_1)}{N\langle \mathbf{p}^2 \rangle} \right)$$

Thus, if there is a first particle with transverse momentum $\mathbf{p}_1$, then the probability to find a second particle with transverse momentum $\mathbf{p}_2$ is NOT isotropic, but larger "away" (in azimuth) from $\mathbf{p}_1$.

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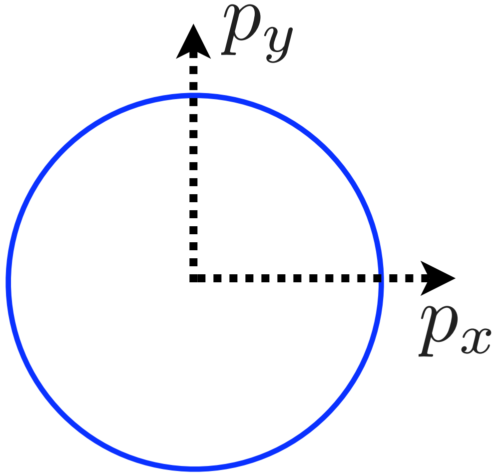

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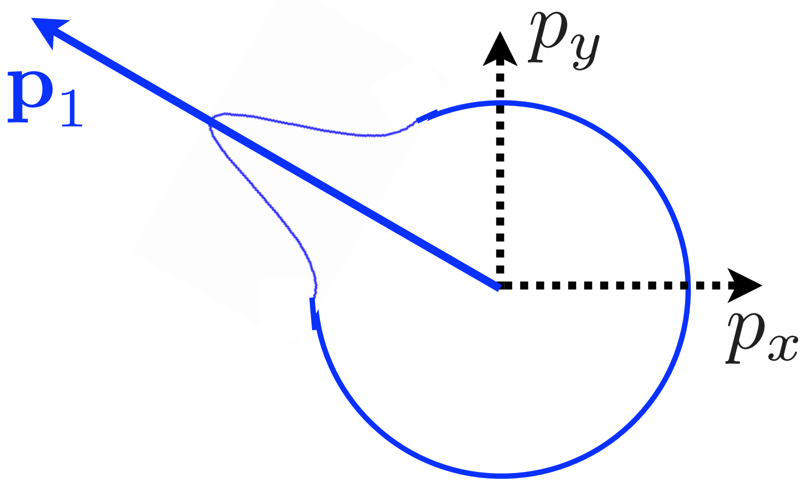

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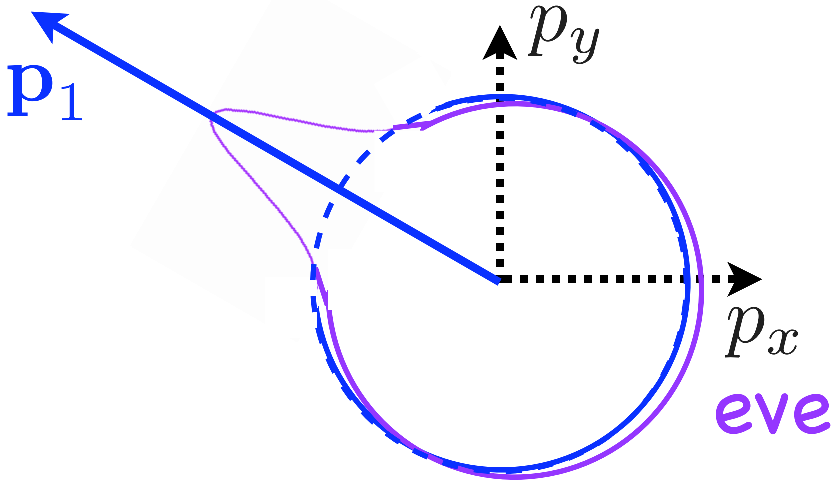


everything recoils

One cannot speak of "a jet + an (uncorrelated) background event"!

Two-particle correlation due to total transverse momentum conservation

The conservation of total transverse momentum does correlate all particles in the event together!

$$f_c(\mathbf{p}_1, \mathbf{p}_2) = -\frac{D\,\mathbf{p}_1 \cdot \mathbf{p}_2}{N\langle \mathbf{p}^2 \rangle}$$

The correlation is back-to-back, & larger between particles with larger momenta

☞ should not be forgotten in jet studies...

Its meaning?

That the conditional probability for an "associated" particle to have a momentum $\mathbf{p}_2$ when there is a "trigger" particle with momentum $\mathbf{p}_1$ is not the same as the probability to have a particle with momentum $\mathbf{p}_2$ irrespective of the momenta of the other particles.

The "background" to the jet is modulated by its presence (need to balance the momentum).

Total momentum conservation and statistical studies of jets

The "background" to the jet is modulated by its presence (need to balance the momentum).
This is a model-independent statement! I do not assume any specific micro-/macroscopic picture of the correlation between the jet and the other particles.

☞ issue for methods which decompose the event into jet+background, as they might not be easy to disentangle from each other.

Safer approach (cf. Claude Pruneau!):
✦ measure the cumulants on the one hand;
✦ compute their values due to various sources of correlation on the other hand.

Three-particle correlation due to total transverse momentum conservation

$$f_c(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \underbrace{-\frac{D}{N^2\langle\mathbf{p}^2\rangle}(\cdots)}_{\text{repulsive term}} + \underbrace{\frac{D^2}{N^2\langle\mathbf{p}^2\rangle^2}(\cdots)}_{\text{attractive term}}$$

The attractive term dominates over the first one when all three particles have transverse momenta larger than the rms transverse momentum: relevant case for high-p_T studies!

Let us investigate the behaviour of this cumulant!
(for simplicity, in the case $p_{\rm trigger} \equiv p_1 > p_2 = p_3 \equiv p_{\rm assoc.}$)

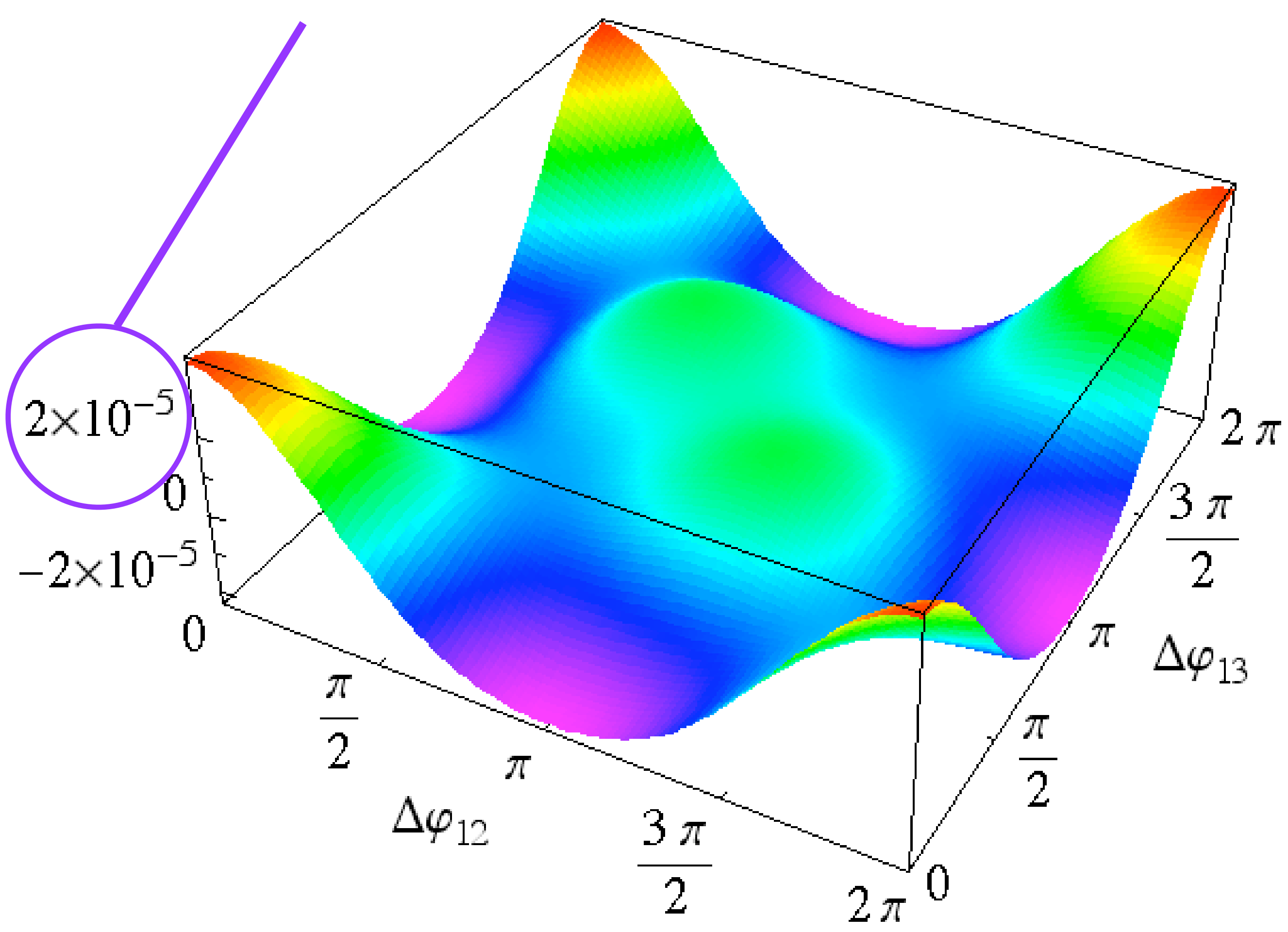
I shall use the relative angles $\Delta\varphi_{12} \equiv \varphi_1 - \varphi_2$ and $\Delta\varphi_{13} \equiv \varphi_1 - \varphi_3$

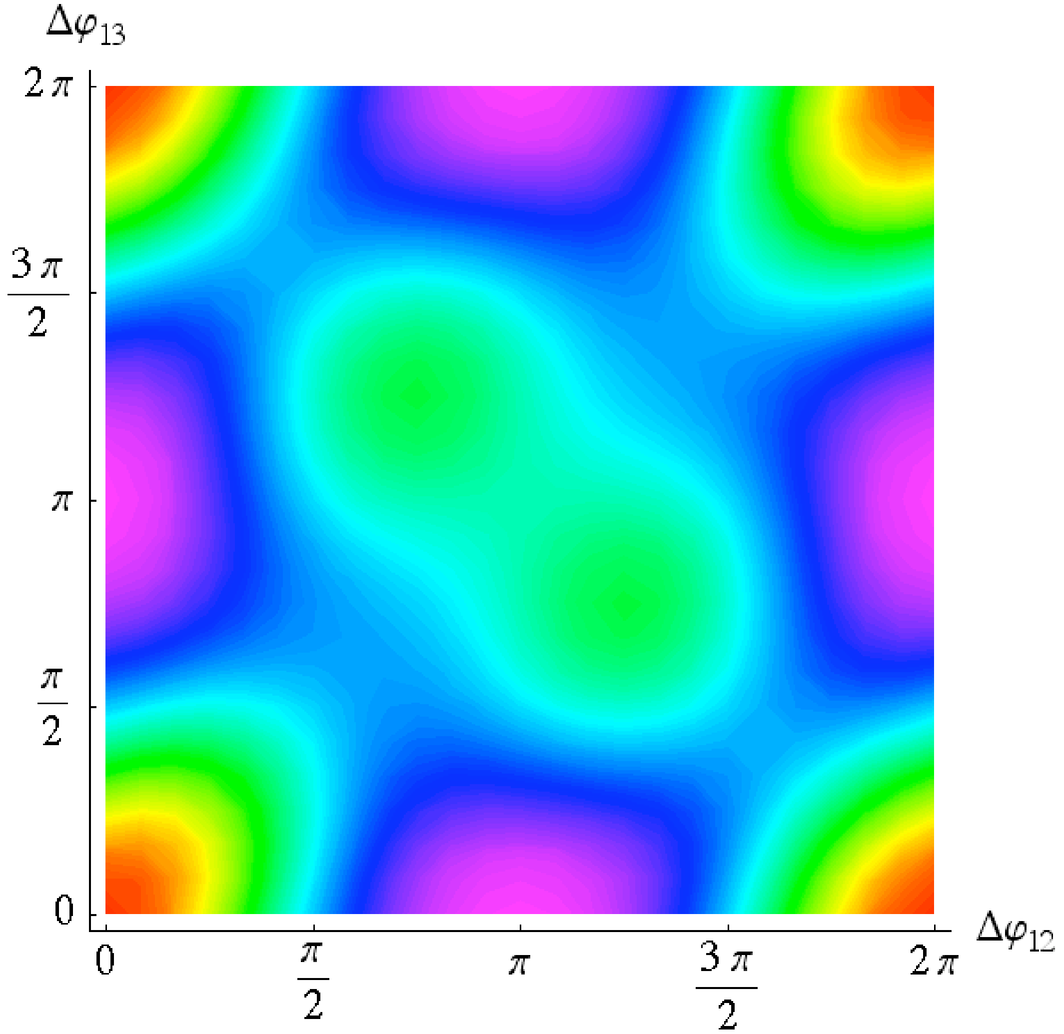
Three-particle correlation due to total transverse momentum conservation

RHIC-inspired values: N = 8000 particles, $\langle \mathbf{p}^2 \rangle^{1/2} = 0.45$ GeV

$p_1 = 3.2$ GeV, $p_2 = p_3 = 1.2$ GeV

same size as correlations due to flow





Note the 2 humps around 180°

Three-particle correlation due to total transverse momentum conservation

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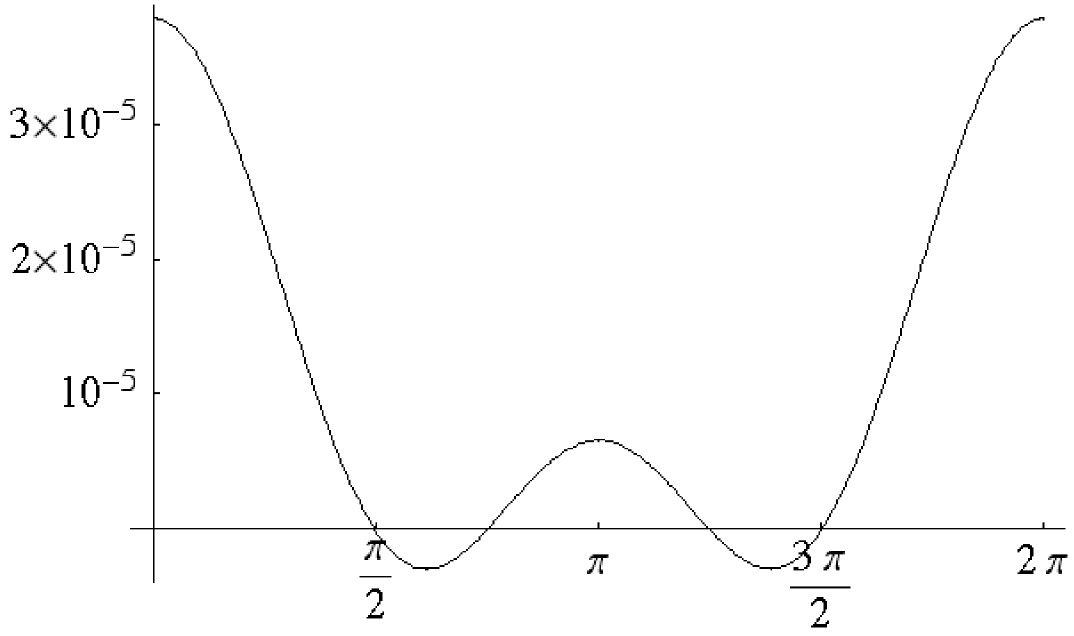
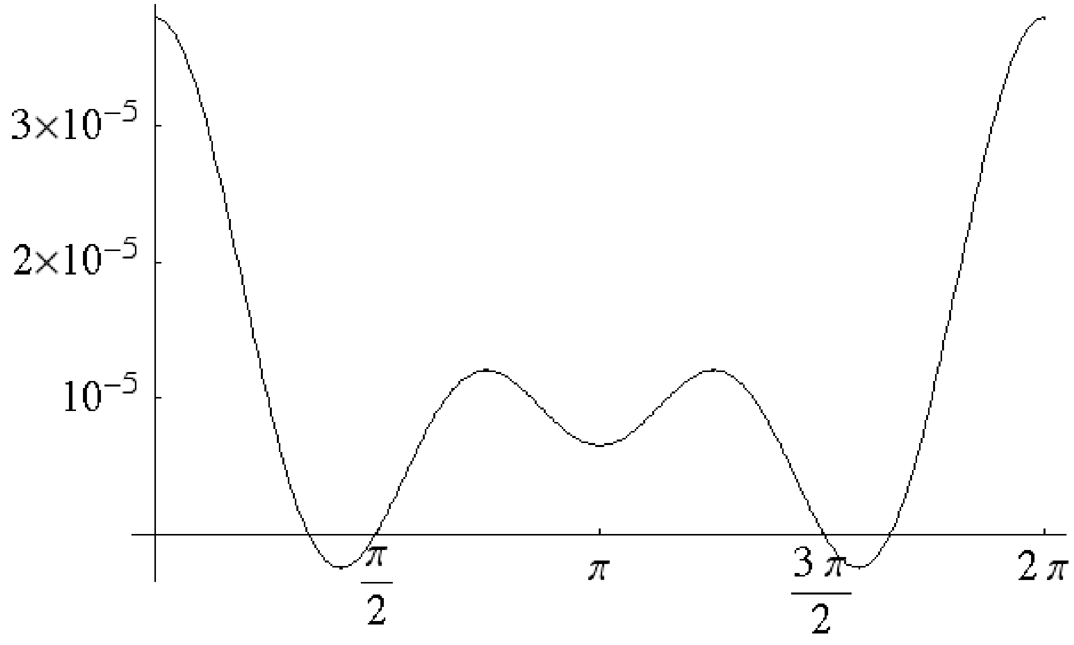
$\Delta\varphi_{12} = \Delta\varphi_{13}$



(Local) maximum at 180°

$\Delta\varphi_{12} = \pi - \Delta\varphi_{13}$



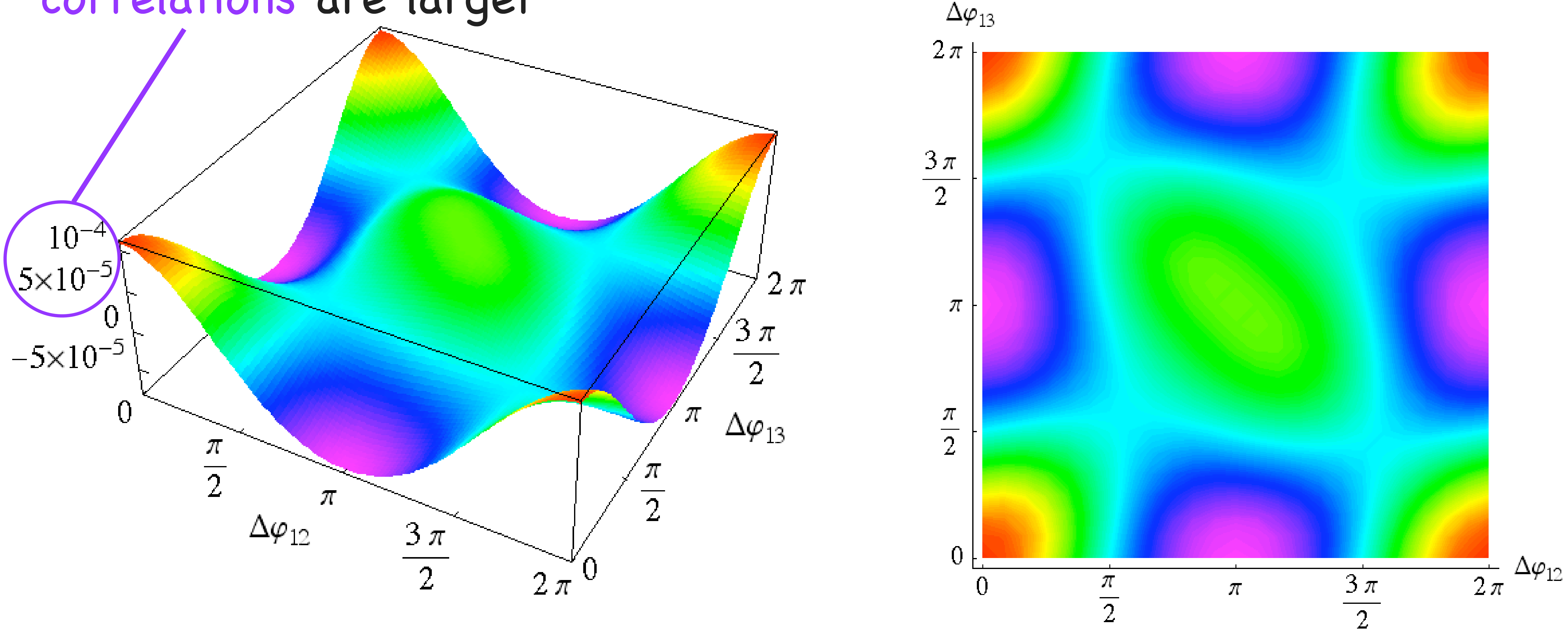
Two humps around 180°

Three-particle correlation due to total transverse momentum conservation

RHIC-inspired values: N = 8000 particles, $\langle \mathbf{p}^2 \rangle^{1/2} = 0.45$ GeV

$p_1 = 6$ GeV $\geq 5(p_2 = p_3) = 1.2$ GeV

correlations are larger



The 2 humps around 180° have merged into one

Three-particle correlation due to total transverse momentum conservation

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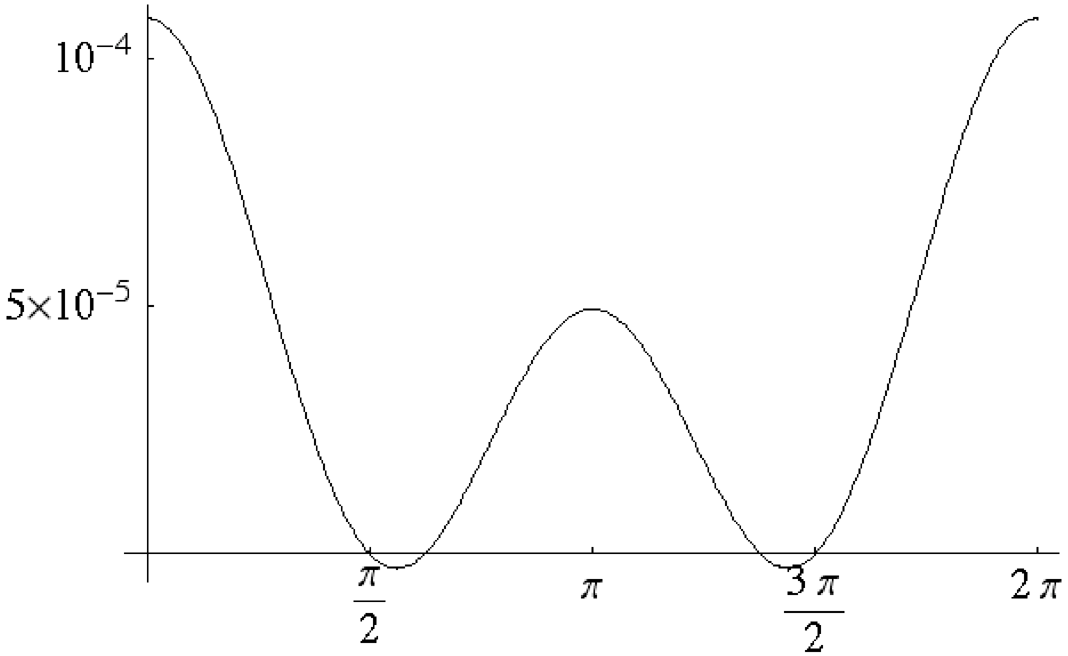
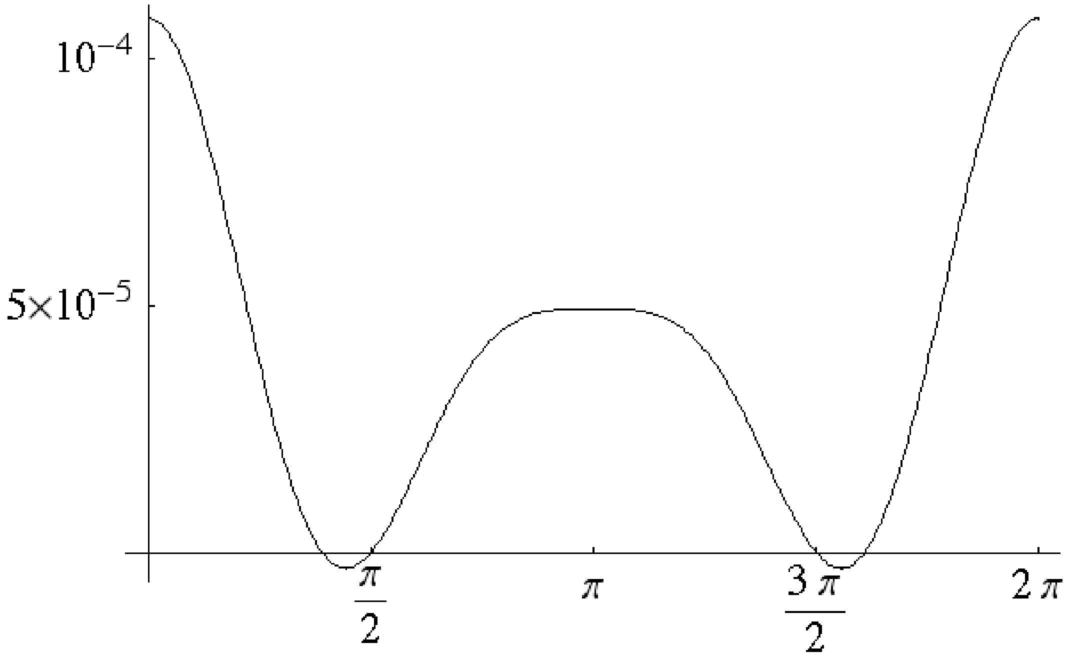




(Local) maximum at 180°

One big hump at 180°

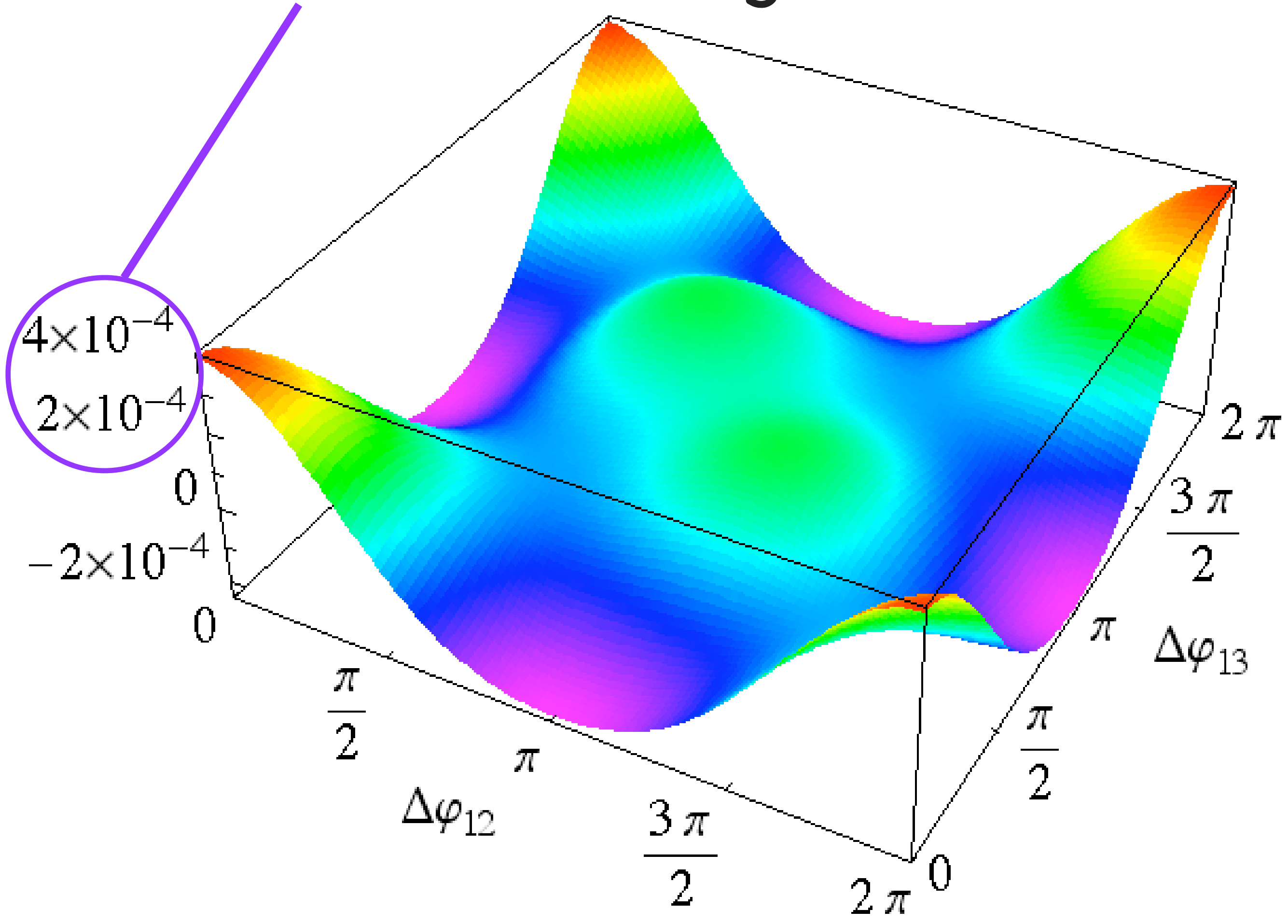
The structure away from the "trigger" depends on the cuts

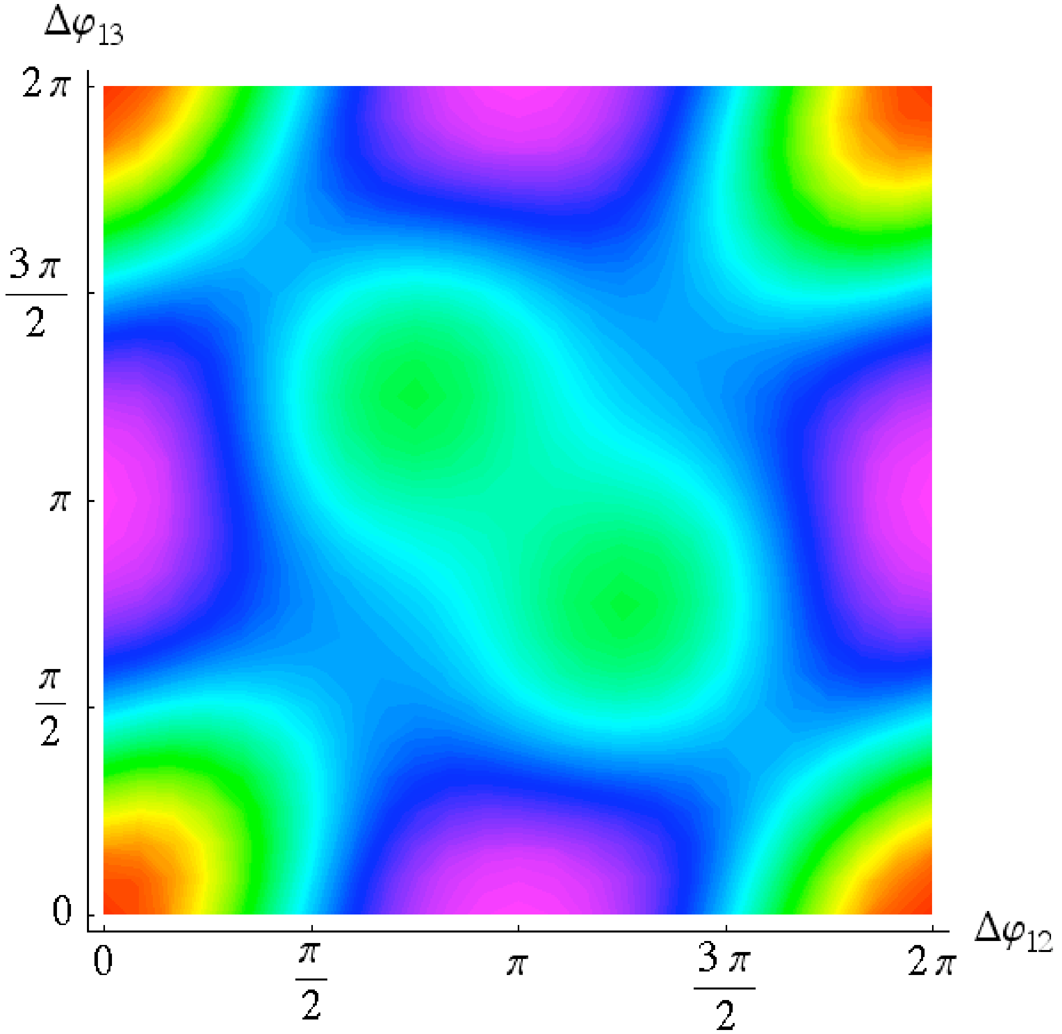
Three-particle correlation due to total transverse momentum conservation

SPS-inspired values: N = 2500 particles, $\langle \mathbf{p}^2 \rangle^{1/2} = 0.45$ GeV

$p_1 = 3.2$ GeV, $p_2 = p_3 = 1.2$ GeV

correlations are larger than at RHIC





Total momentum conservation and statistical studies of jets

Total momentum conservation induces correlations between the particles emitted in a collision.

These correlations can be computed... and their value can be estimated if one “knows” the total emitted multiplicity N and the mean square momentum $\langle \mathbf{p}^2 \rangle$.

☞ can be treated as parameters

Do not underestimate its possible role!