8.9 HTLs and kinetic equations

For the dimensionally reduced theory we would write an effective Lagrangian, e.g.,

(*) $$\mathcal{L}_{3d} = \frac{1}{2} A^0 (-\Delta + m_D^2) A^0$$

with the Debye mass m_D, which came from a HTL.
Can we also write down an effective action which accounts for the real-time HTL?

In addition to the simple mass term in (*) we also get something like

$$\Pi_{\mu\nu}^{HTL} \sim \int \frac{d\Omega}{4\pi} \frac{v_\mu v_\nu}{v \cdot k} + \dots$$

which means that the HTL is not local.

Free action:

$$S = \frac{1}{2} \int d^4x \; A^\mu (-\partial_\mu \partial_\nu + \eta_{\mu\nu} \partial^2) A^\nu$$

check sign: $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \partial_\mu A_\nu \partial^\mu A^\nu + \dots$ ok

Any resummed propagator follows from the eff. action

$$S_{eff} = \frac{1}{2} \int d^4x \, d^4x' \, A^\mu(x) \left[(-\partial_\mu \partial_\nu + \eta_{\mu\nu} \partial^2) \delta(x - x') - \Pi_{\mu\nu}(x - x') \right] A^\nu(x')$$

check the sign in front of Π:

$$S_{eff} \sim \int A^0 (-\Delta + m_D^2) A^0 + \dots$$ ok

The resulting eq. of motion would be

(*) $$(-\partial_\mu \partial_\nu + \eta_{\mu\nu}\partial^2) A^\nu = \int \Pi_{\mu\nu}(x-x') A^\nu(x') d^4x'$$

non-local

Maxwell's eq

$$(-\partial_\mu \partial_\nu + \eta_{\mu\nu}\partial^2) A^\nu = + J_\mu$$

check sign: electrostatics $J_0 = -\Delta A_0$ ok

Thus the RHS of (*) is a current which is non-local in A^ν.

$$J_\mu(x) = \int d^4x' \, \Pi_{\mu\nu}(x-x') A^\nu(x')$$

in Fourier space: $J_\mu(k) = \Pi_{\mu\nu}(k) A^\nu(k)$

Kinetic (Vlasov) equations

We have obtained Π^{HTL} from QFT. Now we will compute J from a completely different starting point. Consider massless particles with charges q_a in an electromagnetic field.

$\{(\vec{x}, \vec{p})\}$ phase space

$f(x, \vec{p}) = f(t; \vec{x}, \vec{p})$ number of particles per phase space volume

Liouville's theorem: f = const along phase space trajectory,

$$\frac{df}{dt} = \partial_t f + \frac{\partial f}{\partial \vec{x}} \cdot \dot{\vec{x}} + \frac{\partial f}{\partial \vec{p}} \dot{\vec{p}} = 0$$

$$\dot{\vec{x}} = \vec{v} = \frac{\vec{p}}{|\vec{p}|}, \quad \dot{p}^\mu = q_a F^{\mu\nu} v_\nu, \quad v = \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

Current:

$$J^\mu = \sum_a q_a \int \frac{d^3p}{(2\pi)^3} f_a \, v^\mu$$

Vlasov equations

In thermal equilibrium: $f(x,\vec{p}) = \underset{\substack{\uparrow \\ f_B \text{ or } f_F}}{f_{eq}}(|\vec{p}|)$, x - independent

$$F_{\mu\nu} = 0 \,,\ J^\mu = 0$$

Consider small fluctuations around equilibrium
$f_a = f_{eq} + \delta f_a$ and linearize the EOM

$$\frac{\partial f_a}{\partial \vec{p}} \cdot \dot{\vec{p}} \simeq \frac{\partial f_{eq}}{\partial \vec{p}}\, \dot{\vec{p}} = f'_{eq} \underbrace{\frac{\partial |\vec{p}|}{\partial \vec{p}}}_{= \vec{v}}\, \dot{\vec{p}}$$

$$= f'_{eq}\, v^m q_a F^{m\nu} v_\nu \underset{\substack{\uparrow \\ F^{mn} = -F^{nm}}}{=} q_a f'_{eq}\, v^m \underbrace{F^{m0}}_{= \partial^m A^0 - \partial^0 A^m \,\overset{!}{=}\, + E^m}$$

$$= q_a f'_{eq}\, \vec{v} \cdot \vec{E}$$

$$v^\mu \partial_\mu \delta f_a = - q_a f'_{eq}\, \vec{v} \cdot \vec{E}$$

$$J^\mu = \sum_a q_a \int \frac{d^3 p}{(2\pi)^3} \delta f_a v^\mu$$

Solve the Vlasov eqs by Fourier (better would be Laplace) transformation

$$-i v \cdot k \ \delta f_a(k, \vec{p}) = - q_a f'_{eq}\, \vec{v} \cdot \vec{E}(k) \quad \Rightarrow$$

$$\delta f_a = - q_a f'_{eq} \frac{i}{v \cdot k}\, \vec{v} \cdot \vec{E}$$

current:

$$J^\mu = -\sum_a q_a^2 \int \frac{d^3p}{(2\pi)^3} f'_{eq} \frac{i}{v\cdot k} v^\mu \, \vec{v}\cdot\vec{E}(k)$$

like in the computation of $\Pi_{\mu\nu}$, the integration over $|\vec{p}|$ factorizes from the angular integration $d\Omega$ over $\vec{v} = \hat{p}$, $\int \frac{d^3p}{(2\pi)^3} f'_{eq} (\ldots) = \int \frac{d\Omega}{4\pi} (\ldots) \left(\int \frac{d^3p}{(2\pi)^3} f'_{eq} \right)$

$$\int \frac{d^3p}{(2\pi)^3} f'_{eq} = (2\pi)^{-3} \int d\Omega \underbrace{\int_0^\infty dp\, p^2 \frac{\partial f_{eq}}{\partial p}}_{= -2 \int_0^\infty dp\, p\, f_{eq}}$$

$$= -2 \int \frac{d^3p}{(2\pi)^3} f_{eq} \frac{1}{p}$$

For $f_{eq} = f_F$ this equals $(-2) \frac{T^2}{24}$

For QED with 1 massless flavor

$$\sum_a q_a^2 = \underset{e^+,e^-}{2} \cdot \underset{\text{spin}}{2}\, e^2$$

$$J_\mu = \underbrace{-4e^2 (-2) \frac{T^2}{24}}_{= \frac{e^2T^2}{3} = m_D^2} \int \frac{d\Omega}{4\pi} \frac{i}{v\cdot k} v_\mu \underbrace{(-v_\nu F^{\nu 0}(k))}_{= -v_\nu (-i)(k^\nu A^0 - k^0 A^\nu)}$$

$$= -m_D^2 \int \frac{d\Omega}{4\pi} \frac{v_\mu}{v\cdot k} (v\cdot k\, A^0 - k^0 v_\nu A^\nu)$$

$$= -m_D^2 \left\{ u_\mu u_\nu - k^0 \int \frac{d\Omega}{4\pi} \frac{v_\mu v_\nu}{v\cdot k} \right\} A^\nu$$

$$\boxed{J_\mu(k) = \Pi^{HTL}_{\mu\nu}(k) A^\nu(k)}$$

That is, the physics of HTLS is that of hard classical charged particles interacting weakly with a classical soft ($k \ll T$) gauge field.

We may write the eqs which determine J as

$$\boxed{J_\mu(x) = m_D^2 \int \frac{d\Omega}{4\pi} W(x, \vec{v})\, v_\mu}$$

where W satisfies

$$\boxed{v \cdot \partial W = \vec{v} \cdot \vec{E}},$$

Since non-abelian gauge fields have the same Π^{HTL} the physics must be very similar as long as additional interactions can be neglected.
This is true also for pure Yang-Mills theory, i.e. with gluons only. Then the particles are the quanta of the hard gauge field modes, while the classical fields are the Fourier modes with $|\vec{k}| \ll T$.