8.5 Computation of $\Delta_{I\bar{I}}$

production rate of sterile neutrinos: $\left(p_0 = \sqrt{\vec{p}^2 + M^2}\right)$

$$(2\pi)^3 2 p_0 \frac{d^3\Gamma}{d^3 p} = - \operatorname{tr}\left\{ \not{p} \left[\Delta^{<}_{I\bar{I}}(p) - \Delta^{>}_{I\bar{I}}(-p) \right] \right\}$$

$$\underset{\substack{\uparrow \\ \text{Sec. } 8.3}}{=} - \operatorname{tr}\left\{ \not{p} \left[- f_F(p^0)\, \rho_{I\bar{I}}(p) - \underbrace{\left(1 - f_F(-p^0)\right)}_{= + f_F(p^0)} \rho_{I\bar{I}}(-p) \right] \right\}$$

$$= f_F(p^0) \operatorname{tr}\left\{ \not{p} \left[\rho_{I\bar{I}}(p) + \rho_{I\bar{I}}(-p) \right] \right\}$$

and

$$\rho_{I\bar{I}}(p) = \frac{1}{i} \operatorname{disc} \Delta_{I\bar{I}}(p)$$

where $\Delta_{I\bar{I}}$ is the analytic continuation of the imaginary-time correlator.

$$I = \sum_\alpha y_\alpha \tilde{\varphi}^\dagger l_\alpha$$

Start in imaginary time:

$$\Delta_{I\bar{I}}(x) = \langle I(x)\, \bar{I}(0) \rangle = \sum_{\alpha\beta} y_\alpha y^*_\beta \langle \tilde{\varphi}^\dagger l_\alpha(x)\, \bar{l}_\beta \tilde{\varphi}(0) \rangle$$

This is a correlation function of SM fields only.

At lowest order in the SM couplings they are treated as free fields. We consider $T > 130$ GeV, so that $\langle \varphi \rangle = 0$, $m_\ell = 0$ (fermion masses vanish) write out the weak-isospin indices

$$\langle \tilde{\varphi}^{\dagger}_i\, \ell_{\alpha i}(x)\, \bar{\ell}_{\beta j}\, \tilde{\varphi}_j(0)\rangle = \underbrace{\langle \tilde{\varphi}^{\dagger}_i(x)\, \tilde{\varphi}_j(0)\rangle}_{=\,\delta_{ij}\,\Delta(x)}\ \underbrace{\langle \ell_{\alpha i}(x)\, \bar{\ell}_{\beta j}(0)\rangle}_{=\,\delta_{ij}\, P_L\, S(x)\, P_R\, \delta_{\alpha\beta}}$$

Scalar

fermion propagator

= 

$\delta_{ij}\delta_{ij} = \delta_{ii} = 2 \quad \Rightarrow$

$P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$
chiral projectors

$$\Delta_{\mathrm{I}\bar{\mathrm{I}}}(x) = 2\sum_{\alpha} |y_\alpha|^2\, \Delta(x)\ P_L\, S(x)\, P_R$$

Fourier transform (still imaginary time, $p^0 = i n \pi T$, n odd)

$$\Delta_{\mathrm{I}\bar{\mathrm{I}}}(p) = 2\sum_{\alpha} |y_\alpha|^2 \int_0^\beta d\tau \int d^3x\, e^{ipx}\ \sum\!\!\!\!\!\!\!\int_q e^{+iqx}\, \Delta(q)$$

$$P_L\ \sum\!\!\!\!\!\!\!\int_k e^{-ikx}\, S(k)\ P_R \qquad\qquad p + q - k = 0$$

$$= 2\sum_{\alpha} |y_\alpha|^2\ \sum\!\!\!\!\!\!\!\int_k \Delta(p-k)\ P_L\, S(k)\, P_R$$

$$= 2\sum_{\alpha} |y_\alpha|^2\, P_L\ \sum\!\!\!\!\!\!\!\int_k \underbrace{\frac{-1}{(k-p)^2}\ \frac{-1}{\not{k}}}_{=\,\frac{1}{(k-p)^2}\,\frac{1}{k^2}\,\not{k}}\ P_R$$

[illegible] we need

$$\mathrm{tr}(\not p \underbrace{P_L \not k P_R}_{= \not k P_R P_R = \not k P_R}) = \frac{1}{2} \mathrm{tr}\left(\not p \not k (1+\gamma^5)\right)$$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^5) = 0$$

check: let $\rho \notin \{\mu, \nu\}$, $(\gamma^\rho)^2 = s \in \{1, -1\}$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^5) = s \ \mathrm{tr}(\gamma^\rho \underbrace{\gamma^\rho \gamma^\mu \gamma^\nu \gamma^5}_{= -\gamma^\mu \gamma^\nu \gamma^5 \gamma^\rho})$$

$$= -s \ \mathrm{tr}(\gamma^\rho \gamma^\rho \gamma^\mu \gamma^\nu \gamma^5) = - \ \mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) \qquad \square$$

$$\Rightarrow \mathrm{tr}(\not p P_L \not k P_R) = \frac{1}{2} \mathrm{tr}(\not p \not k) = 2 p \cdot k$$

complete squares:

$$2 p \cdot k = -\{(k-p)^2 - k^2 - p^2\} \quad \Rightarrow$$

$$\mathrm{tr}\{\not p \Delta_{II}(p)\} = -2 \sum |y|^2 \SumInt_k \left\{ \frac{1}{k^2} - \frac{1}{(k-p)^2} - \frac{p^2}{k^2(k-p)^2} \right\}$$

$$= -2 \sum |y|^2 \left\{ \SumInt_k \frac{1}{k^2} - \SumInt_q \frac{1}{q^2} - p^2 \SumInt_k \frac{1}{k^2 (k-p)^2} \right\}$$

The first two terms are independent of p.

Thus they have no discontinuity and do not contribute to the spectral function.

8.5

Def $\quad E_k := |\vec{k}|$. Use $\frac{1}{ab} = \frac{1}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right)$

$$\frac{1}{k^2(k-p)^2} = \frac{1}{k_0^2 - E_k^2}\,\frac{1}{(k_0-p_0)^2 - E_{k-p}^2}$$

$$= \frac{1}{k_0 - E_k}\,\frac{1}{k_0 + E_k}\,\frac{1}{k_0 - p_0 - E_{k-p}}\,\frac{1}{k_0 - p_0 + E_{k-p}}$$

$$= \frac{1}{2E_k}\,\frac{1}{2E_{k-p}}\left(\frac{1}{k^0 - E_k} - \frac{1}{k_0 + E_k}\right)\left(\frac{1}{k_0 - p_0 - E_{k-p}} - \frac{1}{k_0 - p_0 + E_{k-p}}\right)$$

$$= \frac{1}{-p^0 - E_{k-p} + E_k}\left(\frac{1}{k_0 - E_k} - \frac{1}{k_0 - p_0 - E_{k-p}}\right)$$

$$- \frac{1}{-p_0 + E_{k-p} + E_k}\left(\frac{1}{k_0 - E_k} - \frac{1}{k_0 - p_0 + E_{k-p}}\right) \qquad (*)$$

$$- \frac{1}{-p_0 - E_{k-p} - E_k}\left(\frac{1}{k_0 + E_k} - \frac{1}{k_0 - p_0 - E_{k-p}}\right)$$

$$+ \frac{1}{-p_0 + E_{k-p} - E_k}\left(\frac{1}{k_0 + E_k} - \frac{1}{k_0 - p_0 + E_{k-p}}\right)$$

Now we have only simple poles in k_0 and we can do the Matsubara sum for each term separately

$$T\tilde{\sum_{k^0}} \frac{1}{k^0 - E} = -\left(\frac{1}{2} - f_F(E)\right)$$

$$T\tilde{\sum_{k^0}} \frac{1}{k_0 - p_0 - E} = T\underbrace{\sum_{q^0}}_{\text{bosonic}} \frac{1}{q_0 - E} = -\left(\frac{1}{2} + f_B(E)\right)$$

odd functions of E

These no longer depend on p_0, and the p_0-dependence comes only from the pre-factor

$$\frac{1}{-p^0 + E_1 + E_2}$$

8.5

Now we can analytically continue p_0.

$$\frac{1}{i}\,\mathrm{disc}\,\frac{1}{-p^0+E_1+E_2} = 2\pi\,\delta(p^0-E_1-E_2)$$

where now $\qquad p^0 = \sqrt{\vec{p}^{\,2}+M^2}$

To see which terms can contribute, momentarily go to the rest frame of N, where $\vec{p}=0$.

Only the second term in (*) contributes to

$$\mathrm{tr}\{\not p\,\rho_{\mathcal{I}\bar{\mathcal{I}}}(p)\} = 2\sum|y_\alpha|^2(+p^2)\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2E_k}\,\frac{1}{2E_{k-p}}$$

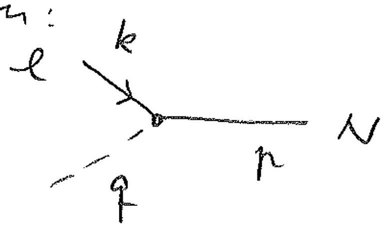
$$(-1)\,2\pi\,\delta(p^0-E_{k-p}-E_k)\Big[f_F(E_k)+\underbrace{f_B(-E_{k-p})}_{=-1-f_B(E_{k-p})}\Big]$$

Exercise :

$$f_F(p_0)\,\mathrm{tr}\{\not p\,\rho_{\mathcal{I}\bar{\mathcal{I}}}(p)\} = 2M^2\sum_\alpha|y_\alpha|^2$$

$$\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2E_k}\,\frac{1}{2E_{k-p}}\,f_F(E_k)\,f_B(E_{p-k})\,2\pi\,\delta(p_0-E_k-E_{k-p})$$

Interpretation:



$$q+k=p$$

"inverse decay"