8.3 Relating correlation functions

H-eigenstate

Start with the Wightman function and use $1 = \sum_m |m\rangle\langle m|$

$$\Delta^>_{AB}(t) = Z^{-1} \mathrm{tr}\Big(e^{-\beta H} \underbrace{A(t)}_{= e^{iHt} A(0) e^{-iHt}} B(0)\Big)$$

$$= Z^{-1} \sum_{mn} e^{(-\beta + it)E_m} \langle m|A(0)|n\rangle e^{-iE_n t} \langle n|B(0)|m\rangle$$

Try to analytically continue to complex t.
This will converge for $-\beta < \mathrm{Im}\, t < 0$

For $t = -i\tau$ this equals our imaginary-time correlator:

(*) $$\Delta_{AB}(-i\tau) = \Delta^>_{AB}(-i\tau) \qquad (\tau \in \mathbb{R})$$

Now consider, for $t \in \mathbb{R}$,

$$\Delta^>_{AB}(t - i\beta) = Z^{-1} \mathrm{tr}\big(e^{+iHt} A(0) e^{-iHt} e^{-\beta H} B(0)\big)$$

$$= \langle B(0) A(t)\rangle = \pm \Delta^<_{AB}(t)$$

Fourier transform

$$\pm \Delta^<_{AB}(\omega) = \int dt\, e^{i\omega t} \Delta^>_{AB}(\overbrace{t - i\beta}^{= t'}) = \int dt'\, e^{i\omega(t' + i\beta)\omega} \Delta^>_{AB}(t')$$

$\Rightarrow$

$$\boxed{\Delta^<_{AB}(\omega) = \pm e^{-\beta\omega} \Delta^>_{AB}(\omega)}$$

Kubo – Martin – Schwinger (KMS) relation

spectral function:

$$\rho_{AB}(t) = \langle [A(t), B(0)]_\mp \rangle = \langle A(t) B(0) \mp B(0) A(t)\rangle$$

$$= \Delta^>_{AB}(t) - \Delta^<_{AB}(t)$$

$\Rightarrow$

$$\rho_{AB}(\omega) = \left(1 \mp e^{-\beta\omega}\right) \Delta^{>}_{AB}(\omega) ,$$

$$\Delta^{>}_{AB}(\omega) = \underbrace{\frac{1}{1 \mp e^{-\beta\omega}}}_{= \frac{e^{\beta\omega}}{e^{\beta\omega} \mp 1} = 1 \pm \frac{1}{e^{\beta\omega} \mp 1}} \rho_{AB}(\omega) \qquad \Rightarrow$$

$$\text{**)} \quad \boxed{\Delta^{>}_{AB}(\omega) = \left(1 \pm f_{B,F}(\omega)\right) \rho_{AB}(\omega)}$$

$$\boxed{\Delta^{<}_{AB}(\omega) = \pm f_{B,F}(\omega) \, \rho_{AB}(\omega)}$$

retarded correlator:

$$\Delta^{ret}_{AB}(t) = i\Theta(t)\,\rho_{AB}(t)$$

$$\Delta^{ret}_{AB}(\omega) = i\int_0^\infty dt\, \rho_{AB}(t)\, e^{i\omega t}$$ analytic in the upper ω-half-plane

Let $\Delta_{AB}(\omega)$ be this analytic continuation. Then for $\omega \in \mathbb{R}$:

$$\boxed{\Delta^{ret}_{AB}(\omega) = \Delta_{AB}(\omega + i0^+)} \left(:= \lim_{\varepsilon \to 0^+} \Delta_{AB}(\omega + i\varepsilon) \right)$$

$$\Delta_{AB}(\omega) = i\int_0^\infty dt\, e^{i\omega t} \int \frac{d\omega'}{2\pi} e^{-i\omega' t} \rho_{AB}(\omega')$$

Interchange the order of integration. The t-integral converges when $\mathrm{Im}\,\omega > 0$.

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$$\Delta_{AB}(\omega) = i \int \frac{d\omega'}{2\pi} \frac{-1}{i(\omega - \omega')} \rho_{AB}(\omega')$$

$$(*) \qquad \boxed{\Delta_{AB}(\omega) = -\int \frac{d\omega'}{2\pi} \frac{\rho_{AB}(\omega')}{\omega - \omega'}}$$

<u>spectral representation</u>

Use this formula to <u>define</u> $\Delta_{AB}(\omega)$ also in the lower half-plane.

Exercise: Show that

$$\boxed{\Delta_{AB}^{adv}(\omega) = \Delta_{AB}(\omega - i0^+)} \qquad (\omega \in \mathbb{R})$$

Now relate this to the imaginary-time correlator

Since $\Delta_{AB}(-i\tau)$ is periodic, we could write it as a Fourier series with Fourier coefficients

$$\tilde{\Delta}_{AB}(i\omega_n) = \int_0^\beta d\tau \, e^{i\omega_n \tau} \Delta_{AB}(-i\tau)$$

Now use (*) and (**) =>

$$\tilde{\Delta}(i\omega_n) = \int_0^\beta d\tau \, e^{i\omega_n\tau} \int \frac{d\omega}{2\pi} e^{-i\omega(-i\tau)} (1 \pm f_{B,F}(\omega)) \rho_{AB}(\omega)$$

$$= \int \frac{d\omega}{2\pi} (1 \pm f_{B,F}(\omega)) \rho_{AB}(\omega) \frac{1}{i\omega_n - \omega} \left(e^{(i\omega_n - \omega)\beta} - 1\right)$$

$$e^{i\omega_n\beta} = \pm 1$$

$$\left(1 \pm f_{B,F}(\omega)\right)\left(e^{(i\omega_n-\omega)\beta} - 1\right)$$
$$= \left(1 \pm \frac{1}{e^{\beta\omega} \mp 1}\right)\left(\pm e^{-\beta\omega} - 1\right)$$
$$= \frac{e^{\beta\omega}}{e^{\beta\omega} \mp 1}\left(\pm e^{-\beta\omega} - 1\right) = -1 .$$

Therefore

$$\tilde{\Delta}_{AB}(i\omega_n) = -\int \frac{d\omega}{2\pi} \frac{\rho_{AB}(\omega)}{i\omega_n - \omega}$$

and

$$\boxed{\tilde{\Delta}_{AB}(i\omega_n) = \Delta_{AB}(i\omega_n)}$$

This means that $\Delta_{AB}(\omega)$ can be obtained by analytic continuation from the discrete set of points $i\omega_n$ on the imaginary axis, thereby giving also all other types of correlation functions discussed so far.

One can show that the analytic continuation is unique if one requires that $\Delta_{AB}(\omega)$ does not grow faster than a power for $\omega \to \infty$.

example: $1 = e^{i\omega_n\beta}$ (bosonic)

unique continuation: 1 (not $e^{i\omega\beta}$)

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Therefore we should invert (*) to determine the spectral function.

Def the <u>discontinuity</u> of Δ_{AB} as $(\omega \in \mathbb{R})$

$$\operatorname{disc} \Delta_{AB}(\omega) = \Delta_{AB}(\omega + i0^+) - \Delta_{AB}(\omega - i0^+)$$

$$\frac{1}{\omega + i0^+ - \omega'} - \frac{1}{\omega - i0^+ - \omega'} = -i2\pi\, \delta(\omega - \omega') \qquad \Rightarrow$$

$$\operatorname{disc} \Delta_{AB}(\omega) = (-1)(-i)\, \rho_{AB}(\omega) \qquad \text{or}$$

$$(*) \qquad \boxed{\rho_{AB}(\omega) = \frac{1}{i} \operatorname{disc} \Delta_{AB}(\omega)}$$

This way all real time correlators can be obtained from the ones in imaginary time.

One should mention that there is also the so-called <u>real-time formalism</u> which allows to do computations directly in real time.

As a very simple example, consider the spatial Fourier transform of the free massless scalar propagator,

$$\Delta(x) = \langle \varphi(x)\,\varphi(0)\rangle \qquad (t = -i\tau):$$

$$\Delta(-i\tau, \vec{k}) = \int d^3_x\, e^{-i\vec{k}\vec{x}}\, \Delta(-i\tau, \vec{x})$$

We had

$$\tilde{\Delta}(i\omega_n, \vec{k}) = \frac{1}{\omega_n^2 + \vec{k}^2} \qquad \Rightarrow$$

$$\Delta(\omega, \vec{k}) = \frac{-1}{\omega^2 - \vec{k}^2}$$

$$\Delta^{ret}(\omega, \vec{k}) \qquad \frac{-1}{(\omega + i0^+)^2 - \vec{k}^2}$$

to compute the spectral function, write

$$\Delta(\omega, \vec{k}) = \frac{-1}{(\omega - |\vec{k}|)(\omega + |\vec{k}|)} = \frac{-1}{2|\vec{k}|}\left(\frac{1}{\omega - |\vec{k}|} - \frac{1}{\omega + |\vec{k}|}\right)$$

Use $(*)$ $\Rightarrow$

$$\rho(\omega, \vec{k}) = \frac{-1}{2|\vec{k}|}(-2\pi)\left[\delta(\omega - |\vec{k}|) - \delta(\omega + |\vec{k}|)\right]$$

$$= 2\pi\, \delta(\omega^2 - \vec{k}^2)\, \mathrm{sgn}(\omega)$$

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