8 Real-time correlations

8.1 Correlation functions

So far we considered correlation functions in imaginary time $t = -i\tau$, $0 \le \tau \le \beta$ defined through the path integral

$$(*) \qquad G(x-x') = Z^{-1} \int \mathcal{D}\varphi \, e^{iS[\varphi]} \varphi(x)\varphi(x')$$

What is the operator expression which gives rise to this path integral representation?

For $\tau_1 > \tau_2$:

$$\langle \varphi(x_1)\varphi(x_2)\rangle = Z^{-1} \operatorname{tr}\left(e^{-\beta H} \varphi(x_1)\varphi(x_2)\right)$$

$$= Z^{-1} \operatorname{tr}\left(e^{-\beta H} e^{H\tau_1} \varphi(\vec{x}_1) e^{-H(\tau_1-\tau_2)} \varphi(\vec{x}_2) e^{-H\tau_2}\right)$$

(↑ Schrödinger pic)

Here we can apply our usual derivation of the path integral by inserting many field and canonical-momentum eigenstates to obtain (*).
Therefore, in the operator representation, the operators must be (imaginary-) time ordered

$$G(x-x') = \langle T \varphi(x)\varphi(x')\rangle \qquad \text{where}$$

$$T\, A(t) B(t') := \Theta(\tau-\tau') A(t) B(t') \pm \Theta(\tau'-\tau) B(t') A(t)$$

with the upper (lower) signs for bosonic (fermionic) operators

The imaginary-time correlators have no direct physical interpretation, except when $\tau = \tau'$. Then they represent spatial equal-time correlations.

One can define various real-time correlation functions. We will see that they can all be obtained from imaginary-time correlators by appropriate analytic continuation.

For any field operators $A(t)$, $B(t)$ (like, e.g. $A(t) = \varphi(t, \underline{x})$) one defines the Wightman functions

$$\Delta^{>}_{AB}(t) := \langle A(t) B(0) \rangle$$

$$\Delta^{<}_{AB}(t) := \pm \langle B(0) A(t) \rangle \quad \text{for} \begin{Bmatrix} \text{bosonic} \\ \text{fermionic} \end{Bmatrix} \text{operators}$$

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$$\rho_{AB}(t) := \langle [A(t), B(0)]_\mp \rangle \quad \text{spectral function}$$

$$\Delta^{ret}_{AB}(t) := i\Theta(t)\langle [A(t), B(0)]_\mp \rangle \quad \text{retarded}$$

$$\Delta^{adv}_{AB}(t) := -i\Theta(-t)\langle [A(t), B(0)]_\mp \rangle \quad \text{retarded}$$

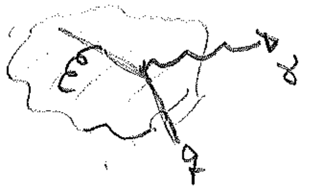
Correlator

Before we discuss relations between these functions we'll consider an application:

8.2 Particle production

typical situations.

(i) photon production in quark-gluon plasma quarks & gluons interact rapidly. When a photon is produced it can most likely escape from the system



(ii) early universe: matter in thermal equilibrium some non-Standard-Model particles like sterile neutrinos, gravitinos, or axion are very weakly coupled. If they are produced they propagate freely

To compute probabilities we need a normalized 1-particle state:

$$|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \Psi(\vec{p}) \; |\vec{p}\rangle$$

$\Psi(\vec{p})$ ↑ momentum space wave-function; $|\vec{p}\rangle$ ↖ 1 particle with momentum $\vec{p}$

$|\vec{p}\rangle = a^\dagger_{\vec{p}} \; |0\rangle$ ↖ vacuum

$$\langle \vec{p} | \vec{p}' \rangle = 2E_p (2\pi)^3 \delta(\vec{p} - \vec{p}')$$

normalization:

$$1 = \langle \Psi | \Psi \rangle = \int \frac{d^3p \, d^3p'}{(2\pi)^6 4E_p E_{p'}} \Psi^*(\vec{p}) \Psi(\vec{p}') \langle \vec{p} | \vec{p}' \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} |\Psi(\vec{p})|^2 \quad \Rightarrow$$

probability to find 1 particle with momentum $\vec{p}$ in d^3p:

$$dP = \frac{1}{2E_p} |\Psi(\vec{p})|^2 \frac{d^3p}{(2\pi)^3} = \frac{1}{2E_p} |\langle \vec{p} | \Psi \rangle|^2 \frac{d^3p}{(2\pi)^3}$$

Now we are ready to deal with particle production.

Let $\mathcal{X}$ be some very weakly coupled field / particle.

Initial condition: no $\mathcal{X}$, but otherwise thermal equilibrium

$\mathcal{L}_{int} = \mathcal{X} \, \mathcal{J}$, $\mathcal{J}$: operator containing the thermal fields

$\mathcal{X}$ creates $\mathcal{X}$-particles

interaction picture ($\mathcal{X} = \bar{\mathcal{X}}$ like photon, Majorana neutrino)

$$\mathcal{X}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left\{ e^{-ipx} a_{\vec{p}} \, u_{\vec{p}} + e^{ipx} a^\dagger_p v \right\}$$

u, v: scalar, spinor or polarization vector time evolution: $i\,|\dot{\Psi}\rangle = H_{int}\,|\Psi\rangle$

$$|\Psi\rangle = \left[1 - i\int_0^t dt'\, H_{int}(t') + \dots\right] |i\rangle \leftarrow \text{initial state}$$

normalized with a single χ particle plus anything:

$$|f\,\vec{p}\rangle = a^\dagger_{\vec{p}}\,|f\rangle$$

$$\langle f\,\vec{p}\,|\Psi\rangle \simeq -i\int_0^t dt_1\, \langle f\,\vec{p}\,|H_{int}(t_1)|i\rangle$$

$$= i\int_0^t dt_1 \int d^3x_1\, \underbrace{\langle f\,\vec{p}\,|\chi\,J|i\rangle}$$

$$\hat{=}\ \langle f|a_p\,\chi\,J(x_1)|i\rangle = e^{ipx_1}\, v\, \langle f|J(x_1)|i\rangle$$

$$|\langle f\,\vec{p}\,|\Psi\rangle|^2 = \int d^4x_1 \int d^4x_2\ \Theta(t_1)\,\Theta(t_2)\, e^{ip(x_1-x_2)}\, v\, v^*$$
$$\langle i|J^\dagger(x_2)|f\rangle\langle f|J(x_1)|i\rangle$$

Sum over all f and thermally average over $|i\rangle$

$$\langle |\langle f\,\vec{p}\,|\Psi|^2\rangle = \int_0^t dt_1 \int d^3x_1 \int_0^t dt_2 \int d^3x_2\, e^{ip(x_1-x_2)}\, v\, v^*\, \underbrace{\langle J^\dagger(x_2)\,J(x_1)\rangle}$$

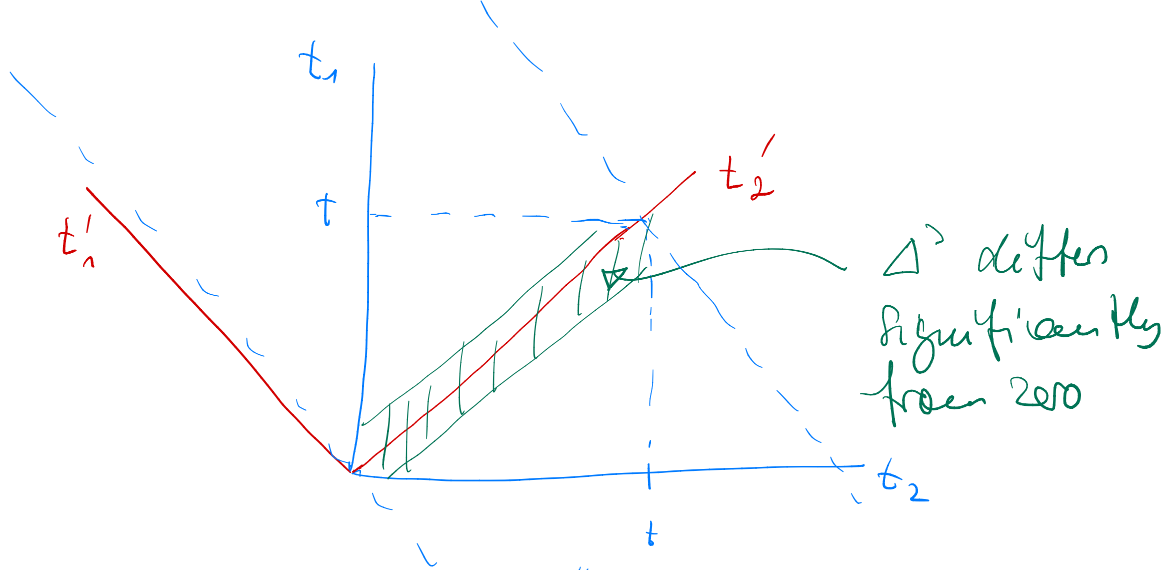
$$\hat{=} \pm\, \Delta^<_{J\,J^\dagger}(x_1 - x_2) \quad \text{for } J: \begin{cases}\text{bosonic}\\ \text{fermionic}\end{cases}$$

Wightman fct.

$$x_1' := x_1 - x_2 \qquad t_2' := t_1 + t_2$$

$$\left|\frac{\partial(t_1', t_2')}{\partial(t_1, t_2)}\right| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

7.1



Since the plasma is thermal, it "forgets" about it's past after a certain time. Therefore $\Delta^>(x_1 - x_2)$ differs significantly from zero only up to a certain time difference $|t_1 - t_2|$. Thus instead of integrating over $0 \le t_1, t_2 \le t$ we may integrate over the much larger region $-\infty < t_1' < \infty$, $0 \le t_2' \le 2t$ without changing the value of the integral:

$$\langle |\langle f \vec{p} | \Phi \rangle|^2 = \pm V \underbrace{\frac{1}{2} \int_0^{2t} dt_2'}_{= Vt} \int dt_1' \int d^3 x_1' \, v v^* \Delta^<(x_1') e^{ipx_1}$$

$\Rightarrow$ production rate / volume

$$d\Gamma = \frac{dP}{Vt} = \pm \frac{d^3 p}{(2\pi)^3 2E_p} \int d^4 x \, v v^* \Delta^<_{JJ^+}(x) e^{ipx}$$

upper / lower sign for bosons / fermions