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6.3 Gluon polarization tensor: gluon self-interactions

Consider non-abelian gauge theory where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$

Choose normalization of generators as

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

$$\mathcal{L}_{gauge} = -\frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}\,F^a_{\mu\nu}F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$\mathcal{L}_{int} = \mathcal{L}_3 + \mathcal{L}_4$$

$$\mathcal{L}_3 = +g\,\partial_\nu A^a_\mu f^{abc} A^{\mu b} A^{\nu c} \qquad \text{cubic}$$

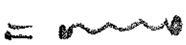
$$\mathcal{L}_4 = -\frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} \qquad \text{quartic}$$

Full gauge field propagator

$$\langle A^a_\mu(x)\, A^b_\nu(0)\rangle =: G^{ab}_{\mu\nu}(x)$$

polarization tensor $\Pi^{ab}_{\mu\nu} = G^{-1\,ab}_{\mu\nu} - \delta^{ab}\Delta^{-1}_{\mu\nu}$

free propagator

$$\Delta^{ab}_{\mu\nu} = \Delta_{\mu\nu}\,\delta^{ab} =$$ 

$\Delta_{\mu\nu}$: free photon propagator

$$G = \Delta - \Delta\Pi\Delta + \dots =$$  $\Rightarrow$

$$-\Pi =$$ 

no external propagators

Π at order g^2

compute $\langle A^a_\mu(x) A^b_\nu(0)\rangle$. Contribution from $\mathcal{L}_4$

 $= \langle A_\mu \, iS_{int} \, A_\nu \rangle_0$

$$= -\frac{g^2}{4} i \int d^{d+1}x' \, f^{cde} f^{cfg} \langle A^a_\mu(x) A^d_\rho A^e_\sigma A^{\rho f} A^{\sigma g}(x') A^b_\nu(0)\rangle_0$$

$$= 4 \langle A^a_\mu(x) A^d_\rho(x')\rangle_0 \left[\langle A^e_\sigma A^{\rho f}\rangle_0 \langle A^{\sigma g}(x') A^b_\nu(0)\rangle_0 + \langle A^e_\sigma A^{\sigma g}\rangle_0 \langle A^{\rho f}(x') A_\nu{}^b(0)\rangle_0 \right]$$

$$= 4 \Delta_{\mu\rho}(x-x') \delta^{ad} \left[\delta^{ef} \Delta_\sigma{}^\rho(0) \delta^{gb} \Delta^\sigma{}_\nu(x') + \delta^{eg} \Delta_\sigma{}^\sigma(0) \delta^{fb} \Delta^\rho{}_\nu(x') \right]$$

Use Feynman gauge ($\xi = 1$)

$\Delta_{\mu\nu} = -\eta_{\mu\nu} \Delta$, Δ = scalar propagator

 $$= -g^2 \int_0^\beta d\tau' \int d^3x' (-1)^3 \Delta(x-x') f^{cae} \underbrace{\left(f^{ceb} \eta_{\mu\sigma} \delta^\sigma_\nu + f^{cbe} (d+1) \eta_{\mu\nu} \right)}_{= f^{cbe} \eta_{\mu\nu} (-1+d+1)} \Delta(0) \Delta(x')$$

$$= -\int_0^\beta d\tau' \int d^3x' \int_0^\beta d\tau'' \int d^3x'' \, \Delta_{\mu\rho}(x-x') \Pi^{\rho\sigma}(x'-x'') \Delta_{\sigma\nu}(x'')$$

$\Rightarrow$

$$\Pi_4^{\rho\sigma}(x) = -d g^2 \eta^{\rho\sigma} f^{cae} f^{cbe} \Delta(0) \delta(\tau) \delta(\vec{x})$$

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$$f^{cae} f^{ceb} = -\left(T_A^c T_A^c\right)^{ab}$$

adjoint representation matrix $(T_A^c)^{ab} = -i f^{abc}$

The matrix $T^c T^c$ commutes with all T^a:

$$[T^a, T^c T^c] = [T^a, T^c] T^c + T^c [T^a, T^c]$$
$$= i f^{acd} \left(T^d T^c + T^c T^d\right) = 0 \quad \Rightarrow$$

$$T_R^c T_R^c = C_R \mathbb{1}$$

for generators in the representation R

C_R is called <u>quadratic Casimir operator</u>

example irreducible representation of angular momentum $\vec{L}$

Casimir operator: $\vec{L}^2 = l(l+1)\mathbb{1}$

Here: $\left(T_A^c T_A^c\right)^{ab} = C_A \delta^{ab} \quad \Rightarrow$

$$\Pi_{4\mu\nu}^{ab}(k) = -g^2 C_A \delta^{ab} \eta_{\mu\nu}\, d\, \Delta(0)$$

independent of k.

For $SU(N)$: $C_A = N$ [see e.g. Peskin, Schroeder 15.4]

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For the contribution from $\mathcal{L}_3$ it is convenient to write it in momentum space and symmetrize it.

$$i\int d_x^{d+1}\mathcal{L}_3 = g\int_0^\beta d\tau \int d_x^d \sum\!\!\!\!\!\!\int_{pqr} \exp(-i[p+q+r]x) f^{abc}$$

$$(-ip_\nu) A_\mu^a(p)\, A^{\mu b}(q)\, A^{\nu c}(r)$$

$$= -ig \sum\!\!\!\!\!\!\int_{pqr} \beta\, \delta_{p_0+q_0+r_0,0}\, (2\pi)^d\, \delta(\vec{p}+\vec{q}+\vec{r})$$

$$f^{abc} A_\mu^a(p)\, A_\nu^b(q)\, A_\rho^c(r)\, p^\rho \eta^{\mu\nu}$$

$$= -ig \sum\!\!\!\!\!\!\int_{pqr} \beta\, \delta_{p_0+q_0+r_0,0}\, (2\pi)^d\, \delta(\vec{p}+\vec{q}+\vec{r})$$

$$\frac{1}{3!} f^{abc} A_\mu^a(p)\, A_\nu^b(q)\, A_\rho^c(r)$$

$$\left[(p-q)^\rho \eta^{\mu\nu} + (q-r)^\mu \eta^{\nu\rho} + (r-p)^\nu \eta^{\rho\mu}\right]$$

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$$= (-ig)^2 \, \frac{1}{2} \left(\frac{1}{3!}\right)^2 \, 2 \cdot 3 \cdot 3 \cdot 2 \; f^{acd} f^{bdc}$$

$$\int_q \left[(k+k+q)^\sigma \eta^{\mu\rho} + (-k-q-q)^\mu \eta^{\rho\sigma} + (q-k)^\rho \eta^{\sigma\mu} \right]$$

$$\frac{1}{q^2 (q+k)^2}$$

$$\left[(-k+q)_\rho \, \eta^\nu{}_\sigma + (-q-k-q)^\nu \eta_{\sigma\rho} + (+k+q+k)_\sigma \eta_\rho{}^\nu \right]$$

$$= -\frac{1}{2} g^2 \left(-f^{cad} f^{cbd}\right) \int_q \frac{1}{q^2 (q+k)^2}$$

$$\Big\{ (2k+q)^\nu (-k+q)^\mu - (2k+q)^\mu (2q+k)^\nu + (2k+q)^2 \eta^{\mu\nu}$$
$$- (2q+k)^\mu (q-k)^\nu + (2q+k)^\mu (2q+k)^\nu (d+1) - (2q+k)^\mu (2k+q)^\nu$$
$$+ (q-k)^2 \eta^{\mu\nu} - (q-k)^\mu (2q+k)^\nu + (q-k)^\nu (2k+q)^\mu \Big\}$$

$$f^{cad} f^{cbd} = C_A \, \delta^{ab}$$

There are 5 tensors that appear in the integrand:

$k^\mu k^\nu$, $k^\nu q^\mu$, $k^\mu q^\nu$, $q^\mu q^\nu$, and $\eta^{\mu\nu}$.

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 $= \frac{g^2}{2} \quad C_A \delta^{ab} \oint_q \frac{1}{q^2(q+k)^2}$

$$\Big\{ k^\mu k^\nu \big(\underbrace{-2 - 2 + 1 + (d+1) - 2 + 1 - 2}_{= d-5} \big)$$

$$+ k^\nu q^\mu \big(\underbrace{2 - 1 + 2 + 2(d+1) - 4 - 1 - 1}_{= 2d-1} \big)$$

$$+ k^\mu q^\nu \big(\underbrace{-1 - 4 - 1 + 2(d+1) - 1 + 2 + 2}_{= 2d-1} \big)$$

$$+ q^\mu q^\nu \big(\underbrace{1 - 2 - 2 + 4(d+1) - 2 - 2 + 1}_{= 4d-2} \big)$$

$$+ \eta^{\mu\nu} \big[k^2 (4+1) + q \cdot k (4-2) + q^2 \, 2 \big] \Big\}$$

$$= 5k^2 + (q+k)^2 - q^2 - k^2 + 2q^2 = 4k^2 + (q+k)^2 + q^2$$

$$= + \frac{g^2}{2} C_A \delta^{ab} \oint_q \frac{1}{q^2(q+k)^2}$$

$$\Big\{ (d-5) k^\mu k^\nu + (2d-1)(k^\mu q^\nu + k^\nu q^\mu) + (4d-2) q^\mu q^\nu$$

$$+ \eta^{\mu\nu} \big(4k^2 + (q+k)^2 + q^2 \big) \Big\}$$

$$\oint_q \frac{q^\nu}{q^2(q+k)^2} \overset{q \to -q-k}{=} \oint_q \frac{1}{(q+k)^2 q^2} (-q^\nu - k^\nu) \quad \Rightarrow$$

move this to the LHS

$$\oint_q \frac{q^\nu}{q^2(q+k)^2} = -\frac{1}{2} \oint_q \frac{k^\nu}{q^2(q+k)^2}$$

$$d - 5 - (2d-1) = -d - 4$$

$\Pi_3^{\mu\nu} = -$  $= -\frac{g^2}{2} C_A \delta^{ab} \oint_q \frac{1}{q^2(q+k)^2} \Big\{ -(4+d) k^\mu k^\nu + (4d-2) q^\mu q^\nu$

$$+ \eta^{\mu\nu} \big(4k^2 + (q+k)^2 + q^2 \big) \Big\}$$

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