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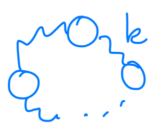
5.4 Pressure at $\mathcal{O}(e^3)$

For the ideal Bose gas we found in the high-T expansion the following contribution $P_{\text{nonanalytic}} = \frac{T m^3}{12\pi}$

In problem H 5.1 we found that the sum of daisy diagrams is obtained by replacing $m^2 \to m^2 + m_{th}^2$ with the thermal mass squared $m_{th}^2 = \Pi$

For the electromagnetic field we have $m=0$, and for A_0 we found the thermal mass $m_{th}^2 = m_D^2$

Thus the resummed $k^0=0$, $|\vec{k}| \ll T$ contribution from



to the pressure is

$$P_{\text{daisy}} = \frac{T}{12\pi}\left(m_D^2\right)^{3/2} = \frac{T^4}{12\pi} e^3 \left(\frac{N_f}{3}\right)^{3/2} \Rightarrow$$

$$\boxed{P = T^4 \left\{ \frac{\pi^2}{45} + \frac{7\pi^2}{180} N_f - \frac{5 N_f}{288} e^2 + \frac{1}{12\pi}\left(\frac{N_f}{3}\right)^{3/2} e^3 + \mathcal{O}(e^4) \right\}}$$

for fermion masses $\ll T$

These corrections are important in cosmology. They determine the equation of state at $T \sim 1$ MeV when neutrinos decouple from the thermal plasma of e^+, e^- and γ. This way they affect N_{eff}, the "effective number of neutrinos".

6. Non-abelian gauge theory

6.1 Gauge transformations

ψ : matter field multiplet

gauge transformation

$$\psi(x) \to U(x)\,\psi(x) \quad , \quad U \text{ unitary}$$

Covariant derivative $\quad D_\mu = \partial_\mu - i g A_\mu$

For

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) - \frac{i}{g}\,\partial_\mu U\, U^\dagger$$

$$(\partial_\mu - i g A_\mu)\psi \to (\partial_\mu - i g\, U A_\mu U^\dagger - \partial_\mu U\, U^\dagger)\, U \psi$$

$$= U \partial_\mu \psi - i g\, U A_\mu \psi$$

I.e. $\quad D_\mu \psi \to U D_\mu \psi$

which can also be written as $\quad D_\mu \to U D_\mu U^\dagger$

Field strength tensor

$$F_{\mu\nu} = \frac{i}{g}\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i g\,[A_\mu, A_\nu]$$

$$F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$$

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Lie Algebra

representation matrices can be written as

$$U = e^{-ig\theta^a T^a} \qquad \theta^a \in \mathbb{R} \qquad T^a: \text{generators, which satisfy ,}$$

$$(*) \qquad [T^a, T^b] = i f^{abc}\, T^c \qquad \underline{\text{Lie algebra}}$$

f^{abc}: structure constants $\in \mathbb{R}$

example: generators of $SU(2)$

$$T^a = \frac{\sigma^a}{2} \qquad \sigma^a: \text{Pauli matrices}$$

$$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\,\varepsilon^{abc}\,\frac{\sigma^c}{2}$$

A_μ , $F_{\mu\nu}$ are elements of the Lie Algebra and can be written as $A_\mu = A_\mu^a T^a$, $F_{\mu\nu} = F_{\mu\nu}^a T^a$

infinitesimal gauge transformation

$$U = 1 - ig\theta^a T^a$$

$$A_\mu \to U A_\mu U^\dagger - \frac{i}{g}\partial_\mu U U^\dagger$$

$$= A_\mu - ig\,\theta^a \underbrace{[T^a, A_\mu^b T^b]}_{= i A_\mu^b f^{abc} T^c} - \frac{i}{g}(-ig)\,\partial_\mu \theta^c T^c$$

$$\Rightarrow$$

$$(**) \qquad A_\mu^c \to A_\mu^c - \partial_\mu \theta^c + g\,\theta^a A_\mu^b f^{abc}$$

particular representations:

(i) fundamental rep. of a group G of unitary matrices: $U \in G$ is represented by U.

dimension: d_F

(ii) adjoint representation

$$(T_A^a)^{bc} = -i f^{abc}$$

The T_A satisfy $(*)$ due to the Jacobi identity

$$[T^a,[T^b,T^c]] + \text{cyclic permutations} = 0$$

dimension d_A

examples a) $U(1)$: $d_F = 1$, $d_A = 1$

b) $SU(N)$: $d_F = N$, $d_A = N^2 - 1$

covariant derivative in the adjoint representation:

$$D_\mu^{bc} = \delta^{bc}\partial_\mu - ig A_\mu^a (T_A^a)^{bc} = \delta^{bc}\partial_\mu - g A_\mu^a f^{abc}$$

Therefore $(**)$ can be written as

$$A_\mu^c \to A_\mu^c - \partial_\mu \theta^c + g A_\mu^a \underbrace{f^{bac}}_{= -f^{abc} = f^{acb}} \theta^b$$

$$= A_\mu^c - D_\mu^{cb}\theta^b$$

6.2 Fadeev-Popov ghosts

The derivation of the path integral formula works exactly like for abelian gauge theory and gives

$$Z = \int \mathcal{D}A\, e^{iS}\, \delta(G[A]) \det\left(\frac{\delta G[A']}{\delta\theta}\right)$$

$$S = S_{gauge} + S_{matter}$$

$$S_{gauge} = -\frac{1}{4}\int d^4x\, F^{a\mu\nu}F^a_{\mu\nu}$$

With the normalization $tr(T^a T^b) = \frac{1}{2}\delta^{ab}$

$$S_{gauge} = -\frac{1}{2}\int d^4x\, tr(F^{\mu\nu}F_{\mu\nu})$$

G is the gauge fixing functional

A' is the infinitesimally gauge transformed field

$$A^{a\,\prime}_\mu = A^a_\mu - D^{ab}_\mu \theta^b$$

Z does not depend on the choice of G.

Choose

$$G^a = \varphi^a - \partial_\mu A^{a\mu}$$

with a scalar field φ. Now we have

$$G^a[A'] = \varphi^a - \partial_\mu\left(A^{\mu a} - D^{\mu ab}\theta^b\right)$$

and

$$\frac{\delta G^a(x)}{\delta \theta^b(x')} = \partial_\mu D^{\mu ab} \delta(x-x')$$

Here we see a difference compared to QED: The

Fadeev - Popov determinant $\det \frac{\delta G}{\delta \theta}$

now depends on A.

Again, we integrate over φ with the weight

$$\exp\left(-\frac{i}{2\xi}\int d^4x\, \varphi^a \varphi^a\right) \Rightarrow$$

$$Z = \int \mathcal{D}A\, e^{iS_{eff}} \det \frac{\delta G}{\delta \theta}$$

with $S_{eff} = S - \frac{1}{2\xi}\int d^4x\, \partial \cdot A^a\, \partial \cdot A^a$

The determinant can be written as a path integral over Grassmann fields with periodic boundary conditions, the Fadeev - Popov ghosts

$$\boxed{Z = \int \mathcal{D}A\, \mathcal{D}\bar{c}\, \mathcal{D}c\, e^{i(S_{eff} + S_{gh})}}$$

$\mathcal{L}_{gh} = \bar{c}^a\, \partial \cdot D^{ab} c^b$ ghost Lagrangian