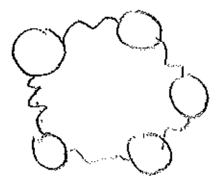
## 5.3 Photon polarization tensor

Since photons are massless, there could be an IR-divergence in

when the photon has $\omega_n = 0$. Then one would have to sum all daisy diagrams



One needs 

This is also part of the full photon propagator

$$G^{\mu\nu}(x) := \langle A^\mu(x) A^\nu(0) \rangle$$

without interaction: $G^{\mu\nu} = \Delta^{\mu\nu}$

For scalars we had

$$G(k) = \frac{1}{-k^2 + m^2 + \Pi} \iff G^{-1} = \Delta^{-1} + \Pi$$

now define the polarization tensor

$$\Pi_{\mu\nu} := G^{-1}_{\mu\nu} - \Delta^{-1}_{\mu\nu}$$

where the inverse propagator is defined through

$$G^{\mu\nu} G^{-1}_{\nu\rho} = \delta^\mu{}_\rho$$

compute $\Pi$ at order $e^2$

$$G = (\Delta^{-1} + \Pi)^{-1} = \left[\Delta^{-1}(1 + \Delta\Pi)\right]^{-1}$$

(*) $$= (1+\Delta\Pi)^{-1}\Delta = \Delta - \Delta\Pi\Delta + \dots$$

$$G^{\mu\nu}(k) = \int_0^\beta d\tau \int d^d x \, e^{ikx} \underbrace{\langle A^\mu(x) A^\nu(0)\rangle}$$

$$= \Delta^{\mu\nu}(k) + \int_0^\beta d\tau \int d^d x \, e^{ikx} \langle A^\mu(x) \tfrac{1}{2}(iS_{int})^2 A^\nu(0)\rangle_0 + \dots$$

$$G^{\mu\nu} - \Delta^{\mu\nu} = -\frac{1}{2}\int d^{d+1}x_1 \int d^{d+1}x_2 \langle A^\mu(x) A^\rho(x_1) J_\rho(x_1) A^\sigma(x_2) J_\sigma(x_2) A^\nu(0)\rangle_0$$

$$= -\int d^{d+1}x_1 \int d^{d+1}x_2 \underbrace{\langle A^\mu(x) A^\rho(x_1)\rangle_0}_{=\Delta^{\mu\rho}(x-x_1)} \langle J_\rho(x_1) J_\sigma(x_2)\rangle_0 \langle A^\sigma(x_2) A^\nu(0)\rangle_0$$

$x \to x + x_1$

$$= -\Delta^{\mu\rho}(k)(-i)\int_0^\beta d\tau_1 \int d^d x_1 \, e^{ikx_1} \int d^{d+1}x_2 \langle J_\rho(x_1 - x_2) J_\sigma(0)\rangle \Delta^{\sigma\nu}(x_2)$$

$x_1 \to x_1 + x_2$

compare with (*) $\Rightarrow$ at leading order we have

$$\boxed{\Pi_{\rho\sigma}(k) = -\int_0^\beta d\tau \int d^d x \, e^{ikx} \langle J_\rho(x) J_\sigma(0)\rangle}$$

NB: one can show that this holds to all orders !!!

at order $e^2$: (1 flavor $N_f = 1$) assume fermion mass $m \ll T$, neglect $m$

$$\langle J_\rho(x) J_\sigma(0)\rangle_0 = e^2 \langle \bar\Psi \gamma_\rho \Psi(x) \bar\Psi \gamma_\sigma \Psi(0)\rangle$$

$$= -e^2 \, \mathrm{tr}\{S(-x)\gamma_\rho S(x)\gamma_\sigma\} = -e^2 \sum\!\!\!\!\!\!\int_p \sum\!\!\!\!\!\!\int_q e^{i(p-q)x}$$

$$\cdot \underbrace{\mathrm{tr}\{S(p)\gamma_\rho S(q)\gamma_\sigma\}}$$

$$= 2^{\frac{d+1}{2}} \frac{1}{p^2 q^2}(p_\rho q_\sigma - \eta_{\rho\sigma}\, p\cdot q + p_\sigma q_\rho)$$

NB: $\mathrm{tr}\, \mathbb{1} = 2^{\frac{d+1}{2}}$ in even space-time dimensions.

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$$k + p - q = 0 \Rightarrow$$

$$\Pi_{\rho\sigma}(k) = +e^2\, 2^{\frac{d+1}{2}} \oint\!\!\!\!\!\!\sum_p \frac{1}{p^2(p+k)^2}\Big( p_\rho (p+k)_\sigma + p_\sigma (p+k)_\rho - \eta_{\rho\sigma}\, p\cdot(p+k)\Big)$$

This is a symmetric 2nd rank tensor. There are 4 such tensors which can be built out of $\eta$, $k$ and and the 4-velocity of the plasma $u$ ($u^0 = 1$, $\vec{u} = 0$ in the plasma rest frame):

$$k_\rho k_\sigma,\ u_\rho u_\sigma,\ \eta_{\rho\sigma},\ u_\rho k_\sigma + u_\sigma k_\rho$$

Current conservation $\partial^\rho j_\rho = 0 \Rightarrow \boxed{k^\rho \Pi_{\rho\sigma}(k) = 0}$

This leaves only 2 independent tensors which can be chosen as

$$P_t^{ij} = \delta^{ij} - \frac{k^i k^j}{\vec{k}^2}, \quad P_t^{\mu 0} = 0 \qquad \text{transverse projector}$$

$$P_\ell^{\mu\nu} = \frac{k^\mu k^\nu}{k^2} - \eta^{\mu\nu} - P_t^{\mu\nu} \qquad \text{longitudinal projector}$$

properties: $k_\mu P^{\mu\nu}_{t,\ell} = 0$

$$P_t^{mr} P_t^{rs} = P_t^{ms} \Rightarrow$$

$$P_t^{\mu\rho} P_{t\rho\sigma} = -P_t{}^\mu{}_\sigma,$$

$$P_\ell^{\mu\rho} P_{t\rho\sigma} = -\eta^{\mu\rho} P_{t\rho\sigma} + P_t{}^\mu{}_\sigma = 0$$

$$P_\ell^{\mu\rho} P_{\ell\rho\sigma} = P_\ell^{\mu\rho}(-\eta_{\rho\sigma}) = -P_\ell{}^\mu{}_\sigma \Rightarrow$$

$$P_t{}^\mu{}_\mu = -P_t^{mm} = -(\delta^{mm} - 1) = -(d-1)$$

$$P_\ell{}^\mu{}_\mu = 1 - \delta^\mu{}_\mu - P_t{}^\mu{}_\mu = 1 - (d+1) + (d-1) = -1$$

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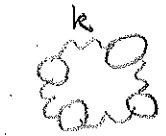
$\Pi^{\mu\nu}$ can be written as

$$\boxed{\Pi^{\mu\nu}(k) = P_t^{\mu\nu}\,\Pi_t + P_\ell^{\mu\nu}\,\Pi_\ell}$$

rotational invariance $\Rightarrow$ $\Pi_{t,\ell}$ only depend on $k^0$ and $|\vec{k}|$.

To compute the $\mathcal{O}(e^3)$ contribution in 
we need, like for scalars, to consider only $k^0 = 0$, $|\vec{k}| \ll T$.

Therefore we compute $\Pi^{\mu\nu}(k)$ in this limit, which simplifies the computation quite a bit.

For $k^0 = 0$:

$P_\ell^{00} = -1$, $P_\ell^{0n} = 0$, $P_\ell^{mn} = 0$ independent of $\vec{k}$.

$\Rightarrow$ the limit $\vec{k} \to 0$ of $P_\ell^{\mu\nu}(0, \vec{k})$ exists.

This limit does not exist for $P_t^{\mu\nu}$. On the other hand, $\Pi^{\mu\nu}(0,\vec{k})$ is finite and has a well defined limit for $\vec{k} \to 0$ $\Rightarrow$ $\boxed{\lim_{\vec{k}\to 0} \Pi_t(0,\vec{k}) = 0}$

The leading contribution to $\Pi^{\mu\nu}(0,\vec{k})$ sits in $\Pi_\ell$.

It can be obtained from

$$\lim_{\vec{k}\to 0} \Pi^\mu{}_\mu(0,\vec{k}) = \lim_{\vec{k}\to 0} P_\ell^{00}\,\Pi_\ell(0,\vec{k}) = \lim_{\vec{k}\to 0} \Pi^{00}(0,\vec{k})$$

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$$\Pi^{\mu}{}_{\mu} = 2^{\frac{d+1}{2}} e^2 \sumint_p \frac{1}{p^2(p+k)^2}\left(2-(d+1)\right) \underbrace{p\cdot(p+k)}$$

$$= p^2 + \tfrac{1}{2}\left[(p+k)^2 - p^2 - k^2\right]$$
$$= \tfrac{1}{2}\left[p^2 + (p+k)^2 - k^2\right]$$

$$= 2^{\frac{d+1}{2}} e^2 (1-d)\,\tfrac{1}{2}\Big\{\underbrace{\sumint_p \frac{1}{(p+k)^2} + \sumint_p \frac{1}{p^2}}_{=\mathcal{O}(T^2)} - \underbrace{k^2 \sumint_p \frac{1}{p^2(p+k)^2}}_{=\mathcal{O}(k^2)}\Big\}$$

For $k^0 = 0$, $|\vec{k}| \ll T$ the third term can be neglected. In this limit

$$\Pi^{\mu}{}_{\mu} = -2^{\frac{d+1}{2}} (d-1) e^2 \underbrace{\sumint_p \frac{1}{p^2}}_{=T^2/24} = -8e^2 \frac{T^2}{24} \Rightarrow$$

$$\Pi^{\mu\nu}(0,\vec{k}) = -\delta^{\mu 0}\delta^{\nu 0}\frac{e^2 T^2}{3} + \mathcal{O}(e^2 \vec{k}^2)$$

For $N_f$ fermion flavors:

$$\boxed{\Pi^{\mu\nu}(0,\vec{k}) = -\delta^{\mu 0}\delta^{\nu 0} m_D^2 + \mathcal{O}(e^2 \vec{k}^2)}$$

$$\boxed{m_D^2 = \tfrac{1}{3} N_f e^2 T^2}$$ Debye mass

Now compute $G^{\mu\nu}$ in this limit.

$$G^{-1}_{\mu\nu} = \Delta^{-1}_{\mu\nu} + \Pi_{\mu\nu}$$

$$\Delta^{-1}_{\mu\nu} = \eta_{\mu\nu} k^2 - \frac{\xi - 1}{\xi} k_\mu k_\nu$$

For $k^0 = 0$: $\Delta^{-1}_{00} = -\vec{k}^2$, $\Delta^{-1}_{0n} = 0$

Inverting $G^{-1}_{\mu\nu}$ is now very easy:

$$G^{00} = (-\vec{k}^2 + \Pi^{00})^{-1}, \quad G^{mn} = \Delta^{mn}$$

for $|\vec{k}| \ll T$:

$$\boxed{G^{00}(0, \vec{k}) = \frac{-1}{\vec{k}^2 + m_D^2}}$$