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4.4 Gauge field propagator

Scalar field: $S_0 = \frac{1}{2}\int d^4x\, \varphi(-\partial^2)\varphi$

propagator in momentum space:

$$\Delta(k) = \frac{-1}{k^2} = (-1) \text{ inverse of the differential operator in } S_0 \text{ (w/o the } 1/2\text{)}$$

gauge field: $Z = \int \mathcal{D}A \exp(iS_{eff})$, $S_{eff} = S - \frac{1}{2\xi}\int d^4x\,(\partial\cdot A)^2$

$$S_0 = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$$

$$= -\frac{1}{4}\int d^4x\,(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{2}\int d^4x\,(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$$

$$= +\frac{1}{2}\int d^4x\,(A_\nu \partial^2 A^\nu - A_\nu \partial_\mu \partial^\nu A^\mu)$$

$$= \frac{1}{2}\int d^4x\, A^\mu(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu$$

$$\int d^4x\, \partial_\mu A^\mu \partial_\nu A^\nu = -\int d^4x\, A^\mu \partial_\mu \partial_\nu A^\nu$$

$$S_{eff,0} = \frac{1}{2}\int d^4x\, A^\mu\left(\eta_{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)A^\nu$$

gauge field propagator: $\Delta^{\mu\nu}(x) = \langle A^\mu(x) A^\nu(0)\rangle_0$

$$\Delta^{\mu\nu}(k) = \left(\eta_{\mu\nu}k^2 - \frac{\xi-1}{\xi}k_\mu k_\nu\right)^{-1}$$

$$\boxed{\Delta^{\mu\nu}(k) = -\frac{1}{k^2}\left(-\eta^{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)}$$

4.4 Check: see problem C6.1

Note that without the gauge fixing term $\frac{1}{\xi}\partial_\mu\partial_\nu$ the differential operator would have a zero-mode,

$$(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\,\partial^\nu \chi = 0$$

and could therefore not be inverted

ξ is a free parameter $\quad \xi = 0$ Landau gauge

$\xi = 1$ Feynman gauge

physical quantities like the pressure must be ξ-independent.

5 Quantum electrodynamics

electromagnetic field interacting with Dirac fermion e.g. electron, muon, ...

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi$$

$$D_\mu = \partial_\mu - ie A_\mu$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = e\bar{\Psi}\not{A}\Psi = -e A_\mu J^\mu$$

5.1 Ideal gas pressure

Once again, computing the ideal gas pressure using the path integral is quite tedious. This is even more so in gauge theories: even in QED one has to include the contribution of the Faddeev - Popov ghosts to get the correct result.

Here we again take the easy route. We know from experience of from the canonical formulation that each gauge field A_μ^a has 2 independent components, corresponding to 2 polarization of, e.g., photons.

Photons are massless. Thus we can use our formula for massless bosons:

$$P_\gamma = 2 \, \frac{\pi^2}{90} T^4 = \frac{\pi^2}{45} T^4$$

Dirac fermions describe particles and antiparticles with spin 1/2. If we have N_f flavors and if we can neglect their masses:

$$P_{\text{fermions}} = \underset{\text{spin}}{2} \cdot \underset{\text{particles + antiparticles}}{2} \, N_f \cdot \frac{7\pi^2}{720} T^4 = \frac{7\pi^2}{180} N_f \, T^4$$

N.B. For Majorana fermions there is only a single factor of 2.

$$P = P_\gamma + P_{\text{fermions}}$$

5.2 Order e^2 corrections

The first corrections to the ideal gas are very similar in QED and in non-abelian gauge theory. However, we start with QED because there are fewer diagrams.

We have $\mathcal{L}_{int} = -e A_\mu J^\mu = e A_\mu \bar{\psi} \gamma^\mu \psi$ and

$$P = P_0 + \frac{T}{V}\left[\langle i S_{int} \rangle_0 + \frac{1}{2} \langle (i S_{int})^2 \rangle_0 - \frac{1}{2} \langle i S_{int} \rangle_0^2 \right] + \mathcal{O}(e^3)$$

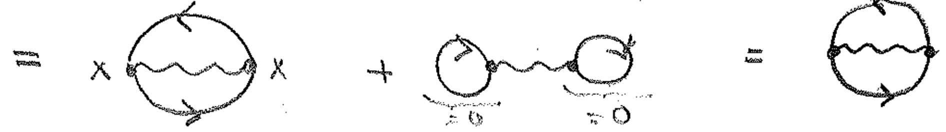
Now $\langle i S_{int} \rangle_0 = \int_0^\beta d\tau \int d^d x \, \underbrace{e \langle A_\mu \bar{\psi} \gamma^\mu \psi \rangle_0}_{= \langle A_\mu \rangle_0 \langle J^\mu \rangle_0}$,

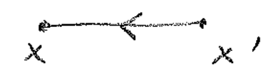
In an electrically neutral plasma the expectation value of the current vanishes, $\langle J^\mu \rangle = 0$. Thus the first correction is

$$\frac{T}{V} \frac{1}{2} \langle (i S_{int})^2 \rangle_0 = \frac{T}{2V} (-e^2) \int d^{d+1} x \int d^{d+1} x' \, \langle A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x) A_\nu(x') \bar{\psi}(x') \gamma^\nu \psi(x') \rangle_0$$

$$= \langle A_\mu(x) A_\nu(x') \rangle \{ \mathrm{tr}\left[-\gamma^\mu \langle \psi(x) \bar{\psi}(x') \rangle_0 \gamma^\nu \langle \psi(x') \bar{\psi}(x) \rangle_0 \right]$$

$$+ \underbrace{\langle \bar{\psi}(x) \gamma^\mu \psi(x) \rangle_0}_{\propto \langle J^\mu \rangle_0 = 0} \underbrace{\langle \bar{\psi}(x') \gamma^\nu \psi(x') \rangle_0}_{= 0} \}$$



$\langle \psi(x) \bar{\psi}(x') \rangle_0 = S(x - x') =$ 

$\langle A_\mu(x) A_\nu(x') \rangle_0 = \Delta_{\mu\nu}(x - x') =$ 

Def $$\sum\!\!\!\!\!\!\!\int_k := T\sum_{k^0 = i2n\pi T} \int \frac{d^dk}{(2\pi)^d}$$

$$\tilde{\sum\!\!\!\!\!\!\!\int_k} := T\sum_{k^0 = i(2n+1)\pi T} \int \frac{d^dk}{(2\pi)^d}$$

Then
$$S(x-x') = \tilde{\sum\!\!\!\!\!\!\!\int_p} e^{-ip(x-x')} \frac{-1}{\not p - m}$$

$$\frac{1}{\not p - m} = \frac{\not p + m}{\not p^2 - m^2} = \frac{\not p + m}{p^2 - m^2}$$

Consider $T \gg m$ and neglect the fermion masses

 $$= \frac{T}{2V} e^2 \int d^{d+1}x \int d^{d+1}x' \sum\!\!\!\!\!\!\!\int_k e^{-ik(x-x')} \Delta_{\mu\nu}(k)$$
$$\mathrm{tr}\left\{\gamma^\mu \tilde{\sum\!\!\!\!\!\!\!\int_p} e^{-ip(x-x')} \frac{-\not p}{p^2} \gamma^\nu \tilde{\sum\!\!\!\!\!\!\!\int_q} e^{-iq(x'-x)} \frac{-\not q}{q^2}\right\}$$
$$= \frac{T}{2V} e^2 \int d^{d+1}x \sum\!\!\!\!\!\!\!\int_k \tilde{\sum\!\!\!\!\!\!\!\int_p} \tilde{\sum\!\!\!\!\!\!\!\int_q} (-i)\beta\, \delta_{k^0+p^0-q^0,0} (2\pi)^d \delta(\vec k + \vec p - \vec q)$$
$$\underbrace{e^{-i(k+p-q)x}}_{\to 1} \underbrace{\mathrm{tr}\{\gamma^\mu \not p \gamma^\nu \not q\}}_{= 4\{p^\mu q^\nu - p\cdot q\, \eta^{\mu\nu} + p^\nu q^\mu\}} \frac{1}{p^2 q^2} \Delta_{\mu\nu}(k)$$

$$= -\frac{e^2}{2} \sum\!\!\!\!\!\!\!\int_k \tilde{\sum\!\!\!\!\!\!\!\int_p} 4\{p^\mu (p+k)^\nu + p^\nu (p+k)^\mu - \eta^{\mu\nu} p\cdot(p+k)\}$$
$$\frac{1}{p^2 (p+k)^2} \Delta_{\mu\nu}(k)$$

Use <u>Feynman gauge</u> $\xi = 1 \to \Delta_{\mu\nu}(k) = \frac{1}{k^2}\eta_{\mu\nu}$

 $= -2e^2 \sum\!\!\!\!\!\!\int_k \sum\!\!\!\!\!\!\int_{\{p\}} \frac{1}{p^2(k+p)^2k^2} \underbrace{\left\{2p\cdot(k+p-(d+1)p\cdot(k+p)\right\}}$

$$-\frac{d-1}{2}(2p\cdot k + 2p^2) = -\frac{d-1}{2}\left((p+k)^2 - p^2 - k^2 + 2p^2\right)$$

$$= -\frac{d-1}{2}(p+k)^2 - p^2 + k^2$$

$$= -(d-1)e^2 \sum\!\!\!\!\!\!\int_k \sum\!\!\!\!\!\!\int_{\{p\}} \left\{-\frac{1}{p^2k^2} - \frac{1}{(k+p)^2k^2} + \frac{1}{p^2(p+k)^2}\right\}$$

$q = k+p$ (fermionic)

$$= -(d-1)e^2\left\{-\sum\!\!\!\!\!\!\int_k \frac{1}{k^2}\sum\!\!\!\!\!\!\int_{\{p\}}\frac{1}{p^2} - \sum\!\!\!\!\!\!\int_k \frac{1}{k^2}\sum\!\!\!\!\!\!\int_{\{q\}}\frac{1}{q^2} + \sum\!\!\!\!\!\!\int_{\{p\}}\frac{1}{p^2}\sum\!\!\!\!\!\!\int_{\{q\}}\frac{1}{q^2}\right\}$$

We had $\displaystyle \sum\!\!\!\!\!\!\int_k \frac{-1}{k^2} = I + I_T \underset{\text{mass}=0}{=} \frac{T^2}{12}$

Similarly $\displaystyle \sum\!\!\!\!\!\!\int_{\{p\}} \frac{-1}{p^2} \underset{\text{zero mass}}{=} -\frac{T^2}{24}$

$d \to 3$

 $\displaystyle = -2e^2 \underbrace{\left\{+2\frac{T^2}{12}\frac{T^2}{24} + \left(\frac{T^2}{24}\right)^2\right\}}_{=\left(\frac{T^2}{12}\right)^2\left(1+\frac{1}{4}\right)}$ for $N_f = 1$ $\Rightarrow$

$$P = T^4\left\{\frac{\pi^2}{45} + \frac{7\pi^2}{180}N_f - \frac{5N_F}{288}e^2 + \mathcal{O}(e^3)\right\}$$

This is similar to the scalar boson gas:

- the first correction is negative, reducing the pressure

[The $\mathcal{O}(e^2)$ corresponds to $\mathcal{O}(\lambda)$]