4.2 Canonical formulation [Weinberg II, ch. 15]

Start with electrodynamics. Our approach can then easily be adopted to non-abelian theories like QCD.

electromagnetic field strength tensor $F^{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) - \rho A^0 + \vec{J}\cdot\vec{A}$$

A^μ: 4-vector potential , $J = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}$ 4-current

A has 4 components, but $\exists$ only 2 photon polarizations.

$S = \int d^4x\, \mathcal{L}$ invariant under gauge transformations

$$A^\mu \to A^\mu + \partial^\mu \chi$$

if the current is conserved, $\partial_\mu J^\mu = 0$.

gauge transform such that

$$A^3 = 0 \qquad \text{"axial gauge"}$$

Canonical momenta: $\pi_\mu = \dfrac{\partial \mathcal{L}}{\partial \dot{A}^\mu}$

$\frac{1}{2}\vec{E}^2 = \frac{1}{2}(\partial_i A_0 + \partial_0 A^i)^2$ contains no $\dot{A}^0$ $\Rightarrow$

$\pi_0 = 0$ "constraint"

also: $\pi_3 = 0$

only 2 non-vanishing canonical momenta

$$\pi_i = \partial_0 A^i + \partial_i A_0 \qquad (i = 1, 2)$$

NB: $\pi_i = -E_i \quad (i = 1, 2)$

Gauß' law: $\nabla \cdot \vec{E} = \rho \Rightarrow -\nabla \cdot \vec{\pi} + \partial_3 E_3 = \rho$ (*)
with $E_3 = -\partial_3 A_0$. This is a constraint for A^0.
It can be solved by x^3-integration. Then A^0 (and E_3) is fixed in terms of $\vec{\pi}$ and ρ, taken at the same time.

Then we have 2 independent fields A^1, A^2 and 2 canonical momenta π_1, π_2.

$$\mathcal{L} = \frac{1}{2}\left[\vec{\pi}^2 + E_3^2 - \vec{B}^2\right] - \rho A^0 + \vec{j} \cdot \vec{A}$$

$$H = \int d^3x \, \{\vec{\pi} \cdot \dot{\vec{A}} - \mathcal{L}\} = \int d^3x \, \{\vec{\pi} \cdot (\vec{\pi} - \nabla A_0) - \mathcal{L}\}$$

$$-\int d^3x \, \vec{\pi} \cdot \nabla A_0 = \int d^3x \, A_0 \nabla \cdot \vec{\pi} = \int d^3x \, A^0 (\partial_3 E_3 - \rho)$$

$$= -\int d^3x \, (E_3 \partial_3 A^0 + A^0 \rho) = \int d^3x \, (E_3^2 - A^0 \rho)$$

$$H = \int d^3x \, \{\tfrac{1}{2}[\vec{\pi}^2 + E_3^2 + \vec{B}^2] - \vec{j} \cdot \vec{A}\}$$

4.3 Path integral

path integral $Z = \int \mathcal{D}\pi_\perp^1 \mathcal{D}\pi_\perp^2 \mathcal{D}A^1 \mathcal{D}A^2 \exp\{ i \int dt \, [-H + \int d^3x \, \vec{\pi}_\perp \dot{\vec{A}}] \}$

problem: exponent is not an integral of a local density because E_3 is a non-local functional.

We have written $\vec{\Pi}_\perp \equiv \begin{pmatrix} \pi^1 \\ \Pi^2 \\ 0 \end{pmatrix}$ because we now

introduce additional the additional integral

$$1 = \int \mathcal{D}\pi_3 \; \delta(\pi_3 + E_3[\vec{\pi}, \rho]) \quad [\pi_3 \text{ is not canonical conjugate of anything}]$$

2

$$\Pi_3 + E_3 = 0 \iff 0 = \partial_3 \Pi_3 + \partial_3 E_3 = \partial_3 \Pi_3 + \rho + \nabla \cdot \vec{\Pi}_\perp$$
$$= \nabla \cdot \vec{\Pi} + \rho$$

For a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$ with one zero $\vec{x}_0$, $\vec{f}(\vec{x}_0) = \vec{0}$ we have

$$\delta(\vec{f}(\vec{x})) = \delta(\vec{x} - \vec{x}_0) \left[\det\left(\frac{\partial f_i}{\partial x_j} \right) \right]^{-1}$$

For our δ-functional:

$$\delta(\Pi_3 + E_3) = \delta(\nabla \cdot \vec{\Pi} + \rho) \det\left(\frac{\delta \nabla \cdot \vec{\Pi}}{\delta \Pi_3} \right)$$

.2

$$\det\left(\frac{\delta\,\nabla\cdot\vec{\pi}}{\delta\,\pi_3}\right) = \det(\partial_3)$$

The δ-functional can also be written as a path integral

$$\delta(\nabla\cdot\vec{\pi}+\rho) = \int \mathcal{D}A^0 \exp\left\{-i\int d^4x\, A^0(\nabla\cdot\vec{\pi}+\rho)\right\}$$

Now we have a local action in the exponent:

$$Z = \int \mathcal{D}\pi_1\,\mathcal{D}\pi_2\,\mathcal{D}\pi_3\,\mathcal{D}A^0\,\mathcal{D}A^1\,\mathcal{D}A^2 \exp\left\{i\int d^4x\left[\vec{\pi}\cdot\dot{\vec{A}} - \tfrac{1}{2}(\vec{\pi}^2+\vec{B}^2)\right.\right.$$
$$\left.\left. + \vec{j}\cdot\vec{A} - A^0(\nabla\cdot\vec{\pi}+\rho)\right]\right\}\det(\partial_3)$$

Integrate out $\vec{\pi}$:

$$\int d^4x\left[-\tfrac{1}{2}\vec{\pi}^2 + \vec{\pi}\cdot\dot{\vec{A}} + \vec{\pi}\cdot\nabla A_0\right]$$
$$= \int d^4x\left[-\tfrac{1}{2}\left(\vec{\pi}-\dot{\vec{A}}-\nabla A_0\right)^2 + \tfrac{1}{2}\left(\dot{\vec{A}}+\nabla A_0\right)^2\right]$$

$$(*)\quad Z = \int \mathcal{D}A^0\,\mathcal{D}A^1\,\mathcal{D}A^2 \exp\left\{i\int d^4x\,\underbrace{\left[\tfrac{1}{2}\left((-\dot{\vec{A}}-\nabla A_0)^2-\vec{B}^2\right)-j_\mu A^\mu\right]}_{=\mathcal{L}}\right\}\det(\partial_3)$$

Now introduce integral over A^3:

$$1 = \int \mathcal{D}A^3\,\delta(A^3)$$

Then Z is of the form

$$\boxed{Z = \int \mathcal{D}A\,\delta(G[A])\,\det\left(\frac{\delta G[A']}{\delta\chi}\right)\exp\left\{i\int d^4x\,\mathcal{L}\right\}}$$

where $A'_\mu = A_\mu \mp \partial_\mu\chi$.

In $(*)$: $G[A] = A_3$, $\det\frac{\delta G}{\delta\chi} = \det\partial_3$

Claim: Z is independent of the gauge fixing functional G.

Faddeev-Popov trick. Also works for non-abelian gauge fields, like in QCD.

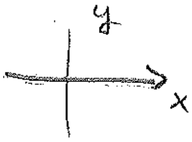
4.2

Check the G independence in a 2-dimensional example:

$a = (x, y)$

$Z = \int dx \, e^{iS}$ where $S = S(x)$

$Z = \int d^2 a \, \delta(y) \, e^{iS(x)}$



Now integrate along this arbitrary curve:

$Z = \int d^2 a \, \delta(y - f(x)) \, e^{iS}$ independent of f;

If this curve is uniquely determined by an implicit relation $G(a) = 0$:

$$Z = \int d^2 a \; \delta(G(a)) \frac{\partial G}{\partial y} \, e^{iS} \text{ , independent of } G$$

We use our freedom to choose

$$G(x) = \phi(x) - \partial_\mu A^\mu(x)$$

with an arbitrary scalar field ϕ. Since this is ϕ-independent, we can integrate over ϕ with the measure

$$\exp\left\{-\frac{i}{2\xi}\int d^4x \, \phi^2\right\}$$

ξ: gauge fixing parameter $\frac{\delta G}{\delta X} = \partial^2$ is an A-independent constant and can be dropped $\Rightarrow$

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}A \, e^{iS_{eff}} \\ S_{eff} &= S - \frac{1}{2\xi}\int d^4x \, (\partial \cdot A)^2 \end{aligned}}$$