3.2 Matsubara sums

antiperiodicity $\Rightarrow$ The Matsubara frequencies in the Fourier series

$$\psi(-i\tau) = T \sum_n e^{-i\omega_n \tau} \psi_n$$

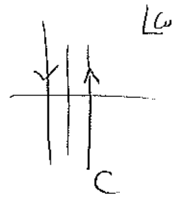
are

$$\boxed{\omega_n = (2n+1)\pi T} \qquad (n \in \mathbb{Z})$$

Sums over Matsubara frequencies:

For bosons we had

$$T \sum_{\omega_n = 2n\pi T} h(i\omega_n) = \int_C \frac{d\omega}{2\pi i} \left[\tfrac{1}{2} + f_B(\omega)\right] h(\omega)$$



$\Rightarrow$

$$T \sum_{\omega_n = (2n+1)\pi T} h(i\omega_n) = T \sum_{\omega_n = 2n\pi T} h(i\omega_n + i\pi T)$$

$$= \int_C \frac{d\omega}{2\pi i} \left[\tfrac{1}{2} + f_B(\omega)\right] h(\omega + i\pi T) = \int_C \frac{d\omega}{2\pi i} \left[\tfrac{1}{2} + f_B(\omega - i\pi T)\right] h(\omega)$$

$$f_B(\omega - i\pi T) = \frac{1}{e^{\beta(\omega - i\pi T)} - 1} = \frac{-1}{e^{\beta\omega} + 1} = -f_F(\omega) \quad \Rightarrow$$

f_F: Fermi distribution

$$\boxed{T \sum_{\substack{\omega_n = (2n+1)\pi T \\ n \in \mathbb{Z}}} h(i\omega_n) = \int_C \frac{d\omega}{2\pi i} \left[\tfrac{1}{2} - f_F(\omega)\right] h(\omega)}$$

3.3 Dirac field

relativistic Dirac fermions (= fermions which are different from their antiparticles) in 4 space-time dimensions are described by 4-component Dirac-Spinor field

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_4(x) \end{pmatrix}$$

action $S = \int d^4x\, \mathcal{L}$, $\quad \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

γ^μ : 4×4 matrices (Dirac matrices) which satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}$$

$\bar{\psi} = \psi^+ \gamma^0$

$$\mathcal{L} = \psi^+ \left(i\left[\partial_t + \gamma^0 \vec{\gamma}\cdot\nabla\right] - \gamma^0 m\right)\psi$$

canonical conjugate to ψ : $\quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^+$

Hamiltonian density

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^+\dot{\psi} - \mathcal{L} = -\bar{\psi}(i\vec{\gamma}\cdot\nabla - m)\psi$$

Hamiltonian: $\quad H = \int d^dx\, \mathcal{H}$

N.B. For $d=0$ we get back to the fermionic harmonic oscillator.

Path integral for the free Dirac field:

$$\boxed{Z = \int\limits_{\substack{\psi(-i\beta,\vec{x}) = -\psi(0,\vec{x}) \\ \bar{\psi}(-i\beta,\vec{x}) = -\bar{\psi}(0,\vec{x})}} \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{iS}, \quad S = \int_0^{-i\beta} dt \int d^3x\, \mathcal{L}}$$

3.3

There are many way for obtaining the 2-point function or (imaginary time) propagator

$$\tilde{S}(x,x') := \langle \psi(x)\bar\psi(x')\rangle = \frac{1}{Z}\int \mathcal{D}\bar\psi\mathcal{D}\psi\, \psi(x)\bar\psi(x')\, e^{iS}$$

Maybe the quickest way is this: $\quad \not\partial := \gamma^\mu\partial_\mu$

$$\frac{\delta S}{\delta\bar\psi(x)} = (i\not\partial - m)\psi$$

NB: The Dirac eqn is $\frac{\delta S}{\delta \bar\psi} = 0$.

$$(i\not\partial - m)\tilde{S}(x,x') = \frac{1}{Z}\int\mathcal{D}\bar\psi\mathcal{D}\psi\, \underbrace{\frac{\delta S}{\delta\bar\psi(x)}e^{iS}}_{=\frac{1}{i}\frac{\delta e^{iS}}{\delta\bar\psi(x)}}\,\bar\psi(x'')$$

assume that we can integrate by parts in the path integral

$$(i\not\partial - m)\tilde{S}(x,x') = \frac{1}{Z}\, i\int\mathcal{D}\bar\psi\mathcal{D}\psi\, e^{iS}\underbrace{\frac{\delta\bar\psi(x')}{\delta\bar\psi(x)}}_{=\delta(x-x')} = i\,\delta(x-x')$$

Fourier transform: $\int_0^\beta d\tau\, e^{i\omega_n\tau}\int d^3x\, e^{-i\vec p\vec x}\,(\dots) \qquad (p^0 = i\omega_n)$

$$(\not p - m)\,S(p,p') = i\int_0^\beta d\tau\, e^{i\omega_n\tau}\int d^3x\, e^{-i\vec p\vec x}$$

$$\underbrace{\int_0^\beta d\tau'\, e^{i\omega_{n'}\tau'}\int d^3x'\, e^{-i\vec p'\vec x'}\; i\,\delta(\tau-\tau')\delta(\vec x-\vec x')}_{=\, i\, e^{i\omega_{n'}\tau}\, e^{-i\vec p'\vec x}} = -\beta\,\delta_{\omega_n+\omega_{n'},0}(2\pi)^3\delta(p-\vec p')$$

$$\Rightarrow \tilde{S}(x,x') = S(x-x'),$$

$$\boxed{S(x-x') = T\sum_{\omega_n=(2n+1)\pi T} e^{-i\omega_n(\tau-\tau')}\int\frac{d^dp}{(2\pi)^d}\, e^{-i\vec p(\vec x-\vec x')}\,\frac{-1}{\not p - m}}$$

3.3 with $p^0 = i\omega_n$ $\qquad$ N.B.: $\langle\psi\psi\rangle = 0$

11/19

Wick's theorem

n - point functions $(n = 2k)$:

$$\langle \psi(x_1) \cdots \psi(x_k) \bar\psi(x_{k+1}) \cdots \bar\psi(x_n) \rangle$$

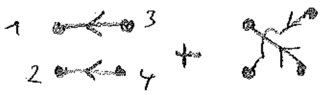
$$= \frac{1}{Z} \int \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{iS} \, \psi(x_1) \cdots \bar\psi(x_n)$$

can be computed using Wick's theorem.

$$\langle \psi(x_1) \cdots \bar\psi(x_n) \rangle = \sum_{\text{Contractions}} \left(\pm \langle \psi(x_1') \bar\psi(x_2') \rangle \cdots \langle \psi(x_{n-1}') \bar\psi(x_n) \rangle \right)$$

where the sign is obtained by moving all $\psi(x_l') \bar\psi(x'_{l+1})$ next to each other.

Example:



$$\langle \psi(x_1) \psi(x_2) \bar\psi(x_3) \bar\psi(x_4) \rangle = - \langle \psi(x_1) \bar\psi(x_3) \rangle \langle \psi(x_2) \bar\psi(x_4) \rangle$$
$$+ \langle \psi(x_1) \bar\psi(x_4) \rangle \langle \psi(x_2) \bar\psi(x_3) \rangle$$

5·3

11/23

4 Gauge fields

We start directly with non-abelian gauge fields like in QCD. Abelian ones like QED can be obtained as a special case.

4.1 Gauge invariance

gauge transformation acting on matter field:

$$\psi(x) \to U(x)\,\psi(x)$$

U: unitary representation of a Lie group G.

covariant derivative

$$D_\mu = \partial_\mu - i g A_\mu \quad , \; A_\mu \in \text{Lie algebra rep.}$$

g: gauge coupling

If $A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$ under gauge transf:

$$D_\mu \psi \to U \psi$$

gauge invariant Dirac-matter Lagrangian:

$$\bar{\psi}(i \not{D} - m)\psi$$

11/23

examples:

(i) QED : $G = U(1)$, ψ 4-component Dirac spinor

$$D_\mu = \partial_\mu - ieA_\mu \qquad U(x) = \exp(-ie\Theta(x))$$

m: particle mass

(ii) QCD $\quad G = SU(3)$

$$\psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^{N_f} \end{pmatrix}$$

$N_f = \#$ flavors $= 6$

$$m = \begin{pmatrix} m_u & & & 0 \\ & m_d & & \\ & & \ddots & \\ 0 & & & m_t \end{pmatrix}$$

$$\psi^\alpha = \begin{pmatrix} \psi_1^\alpha \\ \psi_2^\alpha \\ \psi_3^\alpha \end{pmatrix} \qquad \psi_i^\alpha = \text{4-component Dirac spinor}$$

i = color index $\qquad (U\psi^\alpha)_i = U_{ij}\psi_j^\alpha$

Field strength:

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

gauge transformation:

$$F_{\mu\nu}(x) \to U(x) F_{\mu\nu}(x) U^\dagger(x)$$

gauge invariant gauge field Lagrangian (up to normalization):

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$