3 Fermions

3.1 Path integral

Reminder: path integral in $\underbrace{\text{QM}}$

= bosonic field theory in (0+1) dimensions

$$e^{-iHt} = \underbrace{e^{-iH\Delta t} \cdots e^{-iH\Delta t}}_{N \text{ factors}} \qquad \Delta t = t/N$$

Then we inserted

$$(*) \qquad 1 = \int dx\, |x\rangle\langle x|, \qquad 1 = \int \frac{dp}{2\pi}\, |p\rangle\langle p|$$

between the factors.

N.B. Even for a single bosonic degree of freedom, the Hilbert space is infinite dimensional.

Now consider the

Fermionic harmonic oscillator

= fermionic field theory in (0+1) dimensions

Start from the Lagrangian $\quad L = \psi^+ (i\partial_t - m)\psi.$

canonical momentum $\quad \pi = \frac{\partial L}{\partial \dot\psi} = i\psi^+$

Hamiltonian $\quad H = \pi\dot\psi - L = i\psi^+\dot\psi - \psi^+ i\dot\psi + m\psi^+\psi$

$\overset{!}{=} m\psi^+\psi$

equation of motion $\quad (i\partial_t - m)\psi = 0$

general solution $\quad \psi = e^{-imt} b$

quantization with anticommutators

$\{\psi, \pi\} = i$, $\{\psi,\psi\} = \{\pi,\pi\} = 0 \Leftrightarrow$

$\{b, b^+\} = 1$, $\{b,b\} = \{b^+, b^+\} = 0$

$H = m\, b^+ b$

b, b^+ are annihilation, creation operators

ground state : $|0\rangle$, $b\,|0\rangle = 0$

excited state $|1\rangle = b^+\,|0\rangle$, $b\,|1\rangle = |0\rangle$

$H\,|0\rangle = 0$, $H\,|1\rangle = m\,|1\rangle$

Now the Hilbert space is 2-dimensional.

We want a relation similar to $(*)$.

define Graßmann numbers $\psi, \bar\psi$ through

$\{\psi,\psi\} = \{\psi,\bar\psi\} = \{\bar\psi,\bar\psi\} = 0$

and that they commute with the operators b, b^+.

$\psi, \bar\psi$ are anticommuting c-numbers.

Define the 'coherent states'

$|\psi\rangle = e^{-\bar\psi\psi/2}\,(\,|0\rangle + \psi|1\rangle)$, $\langle\bar\psi| = e^{-\bar\psi\psi/2}(\langle 0| + \langle 1|\bar\psi\,)$

expand the exponentials $\Rightarrow$

$|\psi\rangle = (1 - \bar\psi\psi/2)\,|0\rangle + \psi|1\rangle$, $\langle\bar\psi| = \langle 0|(1-\bar\psi\psi/2) + \langle 1|\bar\psi$

$b\,|\psi\rangle = \psi\,|0\rangle = \psi\,|\psi\rangle$ $|\psi\rangle$ = eigenket of b

$\langle\bar\psi|b^+ = \langle 0|\bar\psi = \langle\bar\psi|\bar\psi$, $\langle\bar\psi|$ = eigenbra of b^+

further properties:

$$\langle \bar\psi_1 | \psi_2 \rangle = \exp\left\{ -\tfrac{1}{2}\bar\psi_1 \psi_1 - \tfrac{1}{2}\bar\psi_2\psi_2 + \bar\psi_1\psi_2 \right\}$$

$$\langle \bar\psi | \psi \rangle = 1$$

$$|\psi\rangle\langle\bar\psi| = (1-\bar\psi\psi)|0\rangle\langle 0| + \bar\psi\,|0\rangle\langle 1| + \psi\,|1\rangle\langle 0| - \bar\psi\psi\,|1\rangle\langle 1|$$

NB: Our notation is a bit misleading, because $|\psi\rangle$ depends on ψ & $\bar\psi$

We would like define the integration over Graßmann variables such that the completeness relation

$$\int d\bar\psi\, d\psi \;\; |\psi\rangle\langle\bar\psi| = 1$$

holds. This is achieved if

$$\int d\psi = \frac{\partial}{\partial\psi}$$, that is derivative = integral:

$$\int d\psi = 0, \quad \int d\psi\; \psi = 1$$

<u>path integral</u>

$$\langle \bar\psi_f | \underbrace{e^{-iH(t_f-t_i)}}_{=\left(e^{-iH\Delta t}\right)^N} |\psi_i\rangle = \quad \text{where} \quad \Delta t = \frac{t_f - t_i}{N}$$

insert $\int d\bar\psi_n\, d\psi_n \; |\psi_n\rangle\langle\bar\psi_n|$ between all factors

$$\langle\bar\psi_n| e^{-iH\Delta t}|\psi_{n-1}\rangle = \langle\bar\psi_n| e^{-imb^+b\,\Delta t}|\psi_{n-1}\rangle$$

$$= \langle\bar\psi_n| 1 - im\, b^+ b\, \Delta t\, |\psi_{n-1}\rangle + O(\Delta t^2) = (1 - im\bar\psi_n\psi_{n-1})\langle\bar\psi_n|\psi_{n-1}\rangle$$

$$= \exp\left\{ -im\bar\psi_n\psi_{n-1}\Delta t - \tfrac{1}{2}\bar\psi_{n-1}\psi_{n-1} - \tfrac{1}{2}\bar\psi_n\psi_n + \bar\psi_n\psi_{n-1} \right\}$$

$$-\frac{1}{2}\left(\bar\psi_{n-1}\psi_{n-1}-\bar\psi_n\psi_{n-1}\right)-\frac{1}{2}\left(\bar\psi_n\psi_n-\bar\psi_n\psi_{n-1}\right)$$

$$=+\frac{1}{2}\left(\bar\psi_n-\bar\psi_{n-1}\right)\psi_{n-1}-\frac{1}{2}\bar\psi_n\left(\psi_n-\psi_{n-1}\right)$$

$$\simeq \Delta t\,\frac{1}{2}\left[\dot{\bar\psi}\psi-\bar\psi\dot\psi\right]=-\Delta t\,\frac{1}{2}\bar\psi\overleftrightarrow{\partial_t}\psi$$

where $\psi_n=\psi(t_n)$

$$\langle\bar\psi_n|e^{-iH\Delta t}|\psi_{n-1}\rangle\simeq\exp\left\{+i\Delta t\,\bar\psi\left(\frac{i}{2}\overleftrightarrow{\partial_t}-m\right)\psi\right\}$$

$$\boxed{\langle\bar\psi_f|e^{-iH(t_f-t_i)}|\psi_i\rangle=\int\mathcal{D}\bar\psi\,\mathcal{D}\psi\,e^{iS}}$$ with

$$S=\int_{t_i}^{t_f}dt\,\bar\psi(t)\left(\frac{i}{2}\overleftrightarrow{\partial_t}-m\right)\psi$$

Boundary conditions: $\bar\psi(t_f)=\bar\psi_f$, $\psi(t_f)=\psi_f$

$\bar\psi(t_i)=\bar\psi_i$, $\psi(t_i)=\psi_i$

What we need for Z is the <u>trace</u> of $\rho = e^{-\beta H}$

Write out

$$\langle \bar\psi_f | \rho | \psi_i \rangle = \left[\langle 0 | (1 - \bar\psi_f \psi_f /2) + \langle 1 | \bar\psi_f \right]$$
$$\rho \left[(1 - \bar\psi_i \psi_i /2) |0\rangle + \psi_i |1\rangle \right]$$

$$= \langle 0|\rho|0\rangle + \psi_i \langle 0|\rho|1\rangle + \bar\psi_f \langle 1|\rho|0\rangle$$
$$-\tfrac{1}{2}(\bar\psi_f \psi_f + \bar\psi_i \psi_i)\langle 0|\rho|0\rangle \quad + \bar\psi_f \psi_i \langle 1|\rho|1\rangle$$
$$-\tfrac{1}{2}\left[\bar\psi_f \psi_f \psi_i \langle 0|\rho|1\rangle + \bar\psi_f \bar\psi_i \psi_i \langle 1|\rho|0\rangle\right]$$
$$+\tfrac{1}{4}\bar\psi_f \psi_f \bar\psi_i \psi_i \langle 0|\rho|0\rangle$$

We need the terms in the second line.
For bosons we had periodic boundary conditions

Try the same here: $\bar\psi_f \overset{?}{=} \bar\psi_i$, $\psi_f \overset{?}{=} \psi_i$ and integrate $\int d\bar\psi_i d\psi_i$

This would give

$$\int d\bar\psi_i d\psi_i \{ - \bar\psi_i \psi_i \langle 0|\rho|0\rangle + \bar\psi_i \psi_i \langle 1|\rho|1\rangle \} = \langle 0|\rho|0\rangle - \langle 1|\rho|1\rangle$$

which is not the trace.

Antiperiodic BC: $\quad \bar\psi_f = -\bar\psi_i$, $\psi_f = -\psi_i \quad \Rightarrow$

$$\int d\bar\psi_i d\psi_i \langle \bar\psi_i | \rho | \psi_i \rangle = \langle 0|\rho|0\rangle + \langle 1|\rho|1\rangle \qquad \underline{\text{ok}} \qquad \Rightarrow$$

$$Z = \mathrm{tr}(e^{-\beta H}) = \int\limits_{\substack{\bar\psi(-i\beta) = -\bar\psi(0) \\ \psi(-i\beta) = -\psi(0)}} \mathcal{D}\bar\psi \mathcal{D}\psi \quad \exp\left\{ i \int_0^{-i\beta} dt\, \bar\psi \left(\tfrac{i}{2}\partial_t - m\right)\psi \right\}$$

integrate by parts, $\bar\psi(-i\beta)\psi(-i\beta) = \bar\psi(0)\psi(0) \Longrightarrow$

$$\boxed{Z = \int\limits_{\substack{\bar\psi(-i\beta) = \bar\psi(0) \\ \psi(-i\beta) = \psi(0)}} \mathcal{D}\bar\psi \mathcal{D}\psi \quad \exp\left\{ i \int_0^{-i\beta} dt\, \bar\psi (i\partial_t - m)\psi \right\}}$$

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