2.7 Infrared resummation

We have seen that the "naive" perturbation theory breaks down when m becomes small.
In this case we can use the high-temperature expansion, and we found

$$P = \frac{\pi^2}{90} T^4 - \frac{1}{24} T^2 m^2 + \frac{1}{12\pi} T m^3 + \mathcal{O}(m^4)$$
$$+ \lambda \left[-\frac{1}{192} T^4 + \frac{1}{32\pi} T^3 m + \mathcal{O}(m^2 T^2) \right]$$
$$+ \lambda^2 \left[\frac{1}{2^9 \pi} \frac{T^5}{m} + \mathcal{O}(T^4) \right] + \mathcal{O}(\lambda^2)$$

Let's have a closer look at the odd powers of m, which are non-analytic in m^2.

$$P_{0,\text{odd}} = + \frac{m^3 T}{12\pi}$$

we obtained from the Matsubara zero-mode contribution to $\frac{d^2 P_0}{d(m^2)^2}$.

$$P_{1,\text{odd}} = \lambda \frac{m T^3}{32\pi}$$

$$P_1 = \infty = -\frac{3}{4} \lambda [I + I_T]^2 = -\frac{3}{4} \lambda I_T^2 + \dots$$

$$I_T = \frac{T^2}{12} - \frac{mT}{4\pi} + \mathcal{O}(m^2)$$

this comes from $\omega_n = 0$

$P_{1,odd}$ came from the cross-term of k zero-mode contribution and the leading term of the high-T expansion of I_+.

This term $\frac{T^2}{12}$ is independent of m, which means that the only scale which enters is T.

In this context the momentum scale $\vec{k} \sim T$ is called <u>hard</u>. Thus $\frac{T^2}{12}$ is from a loop with hard momentum, called <u>hard thermal loop</u> (HTL).

On the other hand, we obtained the term $\left(-\frac{mT}{4\pi}\right)$ from m^2-integration of

$$T\int_{\vec{k}} \frac{1}{(\vec{k}^2+m^2)^2}$$

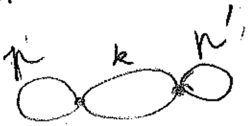
The only scale in this integral is $m \ll T$. Thus $P_{1,odd}$ comes from

with p hard, $k^0=0$, $\vec{k}\sim m$

Finally,

$$P_{2,odd} = \lambda^2 \frac{T^5}{2^9 \pi m}$$

came from



with $k^0=0$, $\vec{k}\sim m$

and p, p' hard.

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What is the expansion parameter controlling the size of $P_{n,odd}$?

$$\frac{P_{1,odd}}{P_{0,odd}} = \frac{3}{8}\,\frac{\lambda T^2}{m^2}$$

$$\frac{P_{2,odd}}{P_{1,odd}} = \frac{1}{24}\,\frac{\lambda T^2}{m^2}$$

I.e., the expansion parameter is something like $\frac{\lambda T^2}{8 m^2}$.

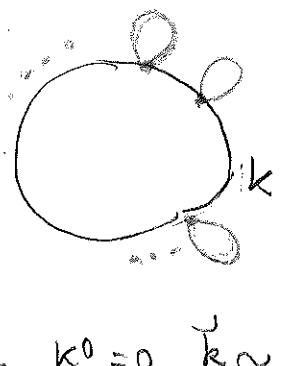
Therefore our expansion will fail not only for $m^2 \to 0$ but already when

$$m^2 \lesssim \frac{\lambda T^2}{8} \qquad !$$

Fortunately, this can be fixed by summing the leading infrared divergent (for $m \to 0$) terms in our loop expansion. This procedure is called infrared resummation.

We can already guess what the leading IR divergent terms at order λ^N are:

HTL



daisy diagram

Gänseblümchen

1 loop with $k^0 = 0$, $\vec{k} \sim m$, N hard thermal loops

We will see in a moment that all daisy diagrams give odd powers of m

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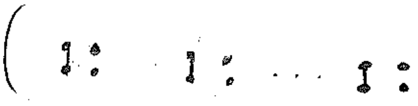
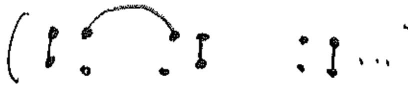
$$P_{N,\text{odd}} = \frac{T}{V}\left\langle \frac{1}{N!}\left(iS_{int}\right)^N\right\rangle + \cdots$$

$$= \frac{T}{V}\frac{(-i)^N}{N!}\int d^{d+1}x_1\cdots\int d^{d+1}x_N\left(-\frac{\lambda}{4}\right)^N \underbrace{\langle\varphi^4(x_1)\cdots\varphi^4(x_N)\rangle_0}_{} + \cdots$$

$= \binom{4}{2}^N$ () $= 6^N(N-1)2$ ()

$= 6^N(N-1)2\cdot(N-2)2$ ()

$$\to 6^N(N-1)!\,2^{N-1}\left[\Delta(0)\right]^N\,\Delta(x_1-x_2)\cdots\Delta(x_N-x_1)$$

$$= \frac{1}{2}\left(\frac{2\cdot 6}{4}\right)^N\frac{(-\lambda)^N}{N}\left[\frac{T^2}{12}\right]^N T\int_{\vec k}\frac{1}{(\vec k^2+m^2)^N}$$

where ... are terms which do not contribute to the leading IR divergence.

$$\int_{\vec k}\frac{1}{(\vec k^2+m^2)^N} = \frac{-1}{N-1}\,\frac{d}{dm^2}\int_{\vec k}\frac{1}{(\vec k^2+m^2)^{N-1}} = \cdots$$

$$= \underbrace{\frac{-1}{N-1}\,\frac{-1}{N-2}\cdots\frac{-1}{2}\,\frac{-1}{1}}_{=\frac{(-1)^{N-1}}{(N-1)!}}\;\frac{d^{N-1}}{d(m^2)^{N-1}}\int_{\vec k}\frac{1}{\vec k^2+m^2}$$

We had

$$\frac{d^2}{d(m^2)^2}P_{0,\text{odd}} = -\frac{1}{2}\,\frac{d}{dm^2}\,T\int_{\vec k}\frac{1}{\vec k^2+m^2} = \frac{T(m^2)^{-1/2}}{16\pi}$$

$$\Rightarrow \int_{\vec k}\frac{1}{\vec k^2+m^2} = -\frac{1}{16\pi}\,2\cdot 2\,(m^2)^{1/2} = -\frac{1}{4\pi}\,\frac{2}{3}\,\frac{d}{dm^2}(m^2)^{3/2}$$

$$= -\frac{1}{6\pi}\,\frac{d}{dm^2}(m^2)^{3/2}$$

$$\int_{\vec{k}} \frac{1}{(\vec{k}^2+m^2)^N} = \frac{(-1)^N}{(N-1)!} \frac{1}{6\pi} \frac{d^N}{d(m^2)^N} m^3 \quad \Rightarrow$$

$$P_{N,\,odd} = \frac{1}{2}\left(\frac{T^2}{4}\right)^N (-\lambda)^N T \; \frac{(-1)^N}{N!} \frac{1}{6\pi} \frac{d^N}{d(m^2)^N} m^3$$

check:

$$P_{0,\,odd} = \frac{1}{2}\frac{1}{6\pi} T m^3 \quad \underline{ok}$$

We can sum all the $P_{N,\,odd}$!

$$\sum_{N=0}^{\infty} P_{N,\,odd} = \frac{T}{12\pi} \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{\lambda T^2}{4}\right)^N \frac{d^N}{d(m^2)^N} (m^2)^{3/2}$$

$$= \frac{T}{12\pi}\left(m^2 + \frac{\lambda T^2}{4}\right)^{3/2}$$

Wow: now one take $m \to 0$ without encountering a divergence !

$$\lim_{m\to 0} \sum_{N=0}^{\infty} P_{N,\,odd} = \frac{T}{12\pi}\left(\frac{\lambda T^2}{4}\right)^{3/2} = \frac{T^4}{12\pi}\frac{\lambda^{3/2}}{8}$$

That is, the next order beyond λ is $\lambda^{3/2}$, not λ^2 !

Thus for $m \to 0$ the perturbative expansion of P is well behaved, and

$$\boxed{P = T^4 \left\{ \frac{\pi^2}{90} - \frac{\lambda}{192} + \frac{\lambda^{3/2}}{96\pi} + \mathcal{O}(\lambda^2) \right\}}$$

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