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2.6 Infrared divergences

We will see that at $O(\lambda^2)$ there are qualitatively new types of contributions to the pressure.

We found that

$$(*) \quad P_2 = \frac{T}{V}\,\frac{1}{2}\left[\,\langle (iS_{int})^2\rangle_0 - \langle S_{int}\rangle_0^2\,\right]$$

Start with the first term $(x^0 = -i\tau)$

$$\frac{T}{V}\langle (iS_{int})^2\rangle = \frac{T}{V}\int_0^\beta d\tau \int d^3x \int_0^\beta d\tau' \int d^3x' \left(\frac{\lambda}{4}\right)^2 \langle \varphi^4(x)\,\varphi^4(x')\rangle_0$$

Wick's theorem gives 3 types of contributions:

(i) every field at x is contracted with a field at x'

$$x \qquad x' \;\to\; 4\,(\;\cdots\;) \;\to\; 4\cdot 3\,(\;\cdots\;)$$

$$\to\; 4\cdot 3\cdot 2\,\left[\Delta(x-x')\right]^4$$

(ii) exactly two fields at x are contracted with fields at x'

$$\longrightarrow\; \underbrace{\binom{4}{2}^2}_{=\left(\frac{4!}{2!2!}\right)^2 = \left(\frac{4\cdot 3\cdot 2}{4}\right)^2 = 36}(\;\cdots\;) = 2\cdot 36\,\left[\Delta(0)\right]^2\left[\Delta(x-x')\right]^2$$

(iii) all fields at x are contracted among themselves

$[$  $]^2$ This is cancelled by the second term in $(*)$.

$$P_2 = \left(\frac{\lambda}{4}\right)^2 \frac{1}{V} \int_0^\beta d\tau \int_0^\beta d\tau' \int d^3x \int d^3x'$$

$$\left\{ 12 \left[\langle \Delta(x-x') \rangle \right]^4 + 36 \left[\Delta(0)\right]^2 \left[\Delta(x-x')\right]^2 \right\}$$

Subst: $x'' = x - x'$

$$\int_0^\beta d\tau \int_0^\beta d\tau' \int d^3x \int d^3x' \left[\Delta(x-x')\right]^n$$

$$= V \int_0^\beta d\tau' \int_{-\tau'}^{\beta-\tau'} d\tau'' \int d^3x'' \left[\Delta(x'')\right]^n$$

The function

$\Delta(x'') = \Delta(-i\tau'', \vec{x}'')$ is β-periodic in τ'' $\Rightarrow$

$$\int_{-\tau'}^{\beta-\tau'} d\tau'' \left[\Delta(x'')\right]^n = \int_0^\beta d\tau'' \left[\Delta(x'')\right]^n$$ independent of τ' $\Rightarrow$

$$P_2 = \left(\frac{\lambda}{4}\right)^2 \int_0^\beta d\tau \int d^3x \left\{ 12 \left[\Delta(x)\right]^4 + 36 \left[\Delta(0)\right]^2 \left[\Delta(x)\right]^2 \right\}$$

$=$  $+$ 

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High T / small m limit

Try to see what happens in the high T or low mass limit $m \ll T$. We have seen that for P_0 and P_1 the limit $m \to 0$ exists.
What about P_2?

Study the behavior of $\Delta(x)$ at short and large distances.

Short distances: we already computed $\Delta(0)$ and found not problem with $m \to 0$.

Long distances We had

$$(*) \quad \Delta(x) = T \sum_n \int_{\vec{k}} e^{-i\omega_n \tau} e^{i\vec{k}\vec{x}} \frac{1}{\omega_n^2 + E_k^2}$$

For $\tau > 0$:

$$\Delta(x) = \int_{\vec{k}} \frac{e^{i\vec{k}\vec{x}}}{2E_k} \frac{\cosh\left(\left[\frac{\beta}{2} - \tau\right] E_k\right)}{\sinh(\beta E_k / 2)}$$

large distances: $|\vec{x}| \gg T^{-1}$. When $|\vec{k}| \gtrsim T$, the integrand oscillates rapidly, giving a small contribution

$\Rightarrow$ dominant contribution from $|\vec{k}| \ll T$

Since also $m \ll T$, we have $\beta E_k \ll 1$ and $\tau E_k \ll 1$, $\tau \in [0, \beta]$

Expand cosh, sinh. $\Rightarrow$

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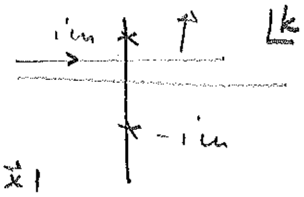
$$\Delta(x) \simeq \int_{\vec k} \frac{e^{i\vec k\vec x}}{2E_k}\,\frac{2T}{E_k} = T\int_{\vec k}\frac{e^{i\vec k\vec x}}{\vec k^2+m^2}$$

NB This is precisely the $\omega_n = 0$ piece of (*).

compute this integral in $d=3$:

$$\Delta(x) \simeq \frac{T}{(2\pi)^2}\int_0^\infty dk\,k^2 \underbrace{\int_{-1}^{1} dc\,\frac{e^{ik|\vec x|c}}{k^2+m^2}}_{=\frac{1}{k^2+m^2}\,\frac{e^{ik|\vec x|}-e^{-ik|\vec x|}}{ik|\vec x|}}$$

$$= \frac{T}{(2\pi)^2}\,\frac{1}{i|\vec x|}\int_{-\infty}^{\infty}\frac{dk\,k\,e^{ik|\vec x|}}{k^2+m^2} =$$



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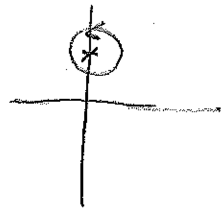

$$(*) \quad = \frac{T}{2\pi}\,\frac{1}{|\vec x|}\,\frac{e^{-m|\vec x|}\,im}{2im} = T\,\frac{e^{-m|\vec x|}}{4\pi|\vec x|}$$

<u>Remark</u>:
Compare this with the $T\to 0$, $m\to 0$ limit of $\Delta(x)$.

For dimensional reasons

$$\Delta(x) \propto \frac{1}{x^2}$$

This falls off much faster at $x^2\to\infty$ than (*).

estimate the long distance contribution to

$$\sim \int d^3x\,\frac{e^{-4m|\vec x|}}{\vec x^4}$$

convergent at large $|\vec x|$ even for $m\to 0$.

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However:  $\propto \int d^3_x \, \frac{e^{-2m|\vec{x}|}}{\vec{x}^2}$

For $m \to 0$ this diverges at $|\vec{x}| \to \infty$; IR-divergence

Compute the contribution to P_2 that is IR divergent for $m \to 0$.

$$P_2 \simeq$$  $$= \left(\frac{\lambda}{4}\right)^2 36 [\Delta(0)]^2 \int_0^\beta d\tau \int d^3_x \, [\Delta(x)]^2$$

In the high-T limit

$$\Delta(0) = J + J_T = \frac{T^2}{12} + \Theta(Tm)$$

For $\Delta(x)$ we use the approximation $(*)$

$$\int_0^\beta d\tau \int d^3_x \, [\Delta(x)]^2 \simeq \beta \left(\frac{T}{4\pi}\right)^2 \int d^3_x \, \frac{e^{-2m|\vec{x}|}}{\vec{x}^2}$$

$$= \frac{T}{4\pi} \int_0^\infty dx \, e^{-2mx} = \frac{T}{4\pi} \frac{1}{2m} = \frac{1}{8\pi} \frac{T}{m}$$

$$P_2 \simeq \left(\frac{6}{4}\right)^2 \lambda^2 \left(\frac{T^2}{12}\right)^2 \frac{1}{8\pi} \frac{T}{m} = \underbrace{\left(\frac{6}{4 \cdot 12}\right)^2 \frac{1}{8\pi}}_{= \frac{1}{8^3 \pi}} \frac{\lambda^2 T}{m}$$

$$\boxed{P_2 = \frac{\lambda^2}{2^9 \pi} \frac{T^5}{m} + \Theta(T^4)}$$ divergent for $m \to 0$

We will see that the same problem arises for photons/gluons in QED/QCD.

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