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2.5 Renormalization at $O(\lambda)$

We had

$$P = P_0 + P_1 + O(\lambda^2) \qquad \text{and}$$

$$P_1 = -\frac{3\lambda_B}{4}\left[I + I_T \right]^2$$

$\uparrow$ UV-finite, T-dependent

$$I = -\frac{m_B^2}{(4\pi)^2}\left\{ \frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{m_B^2}{4\pi} \right\}$$

m_B, λ_B are the parameters appearing in $\mathcal{L}$, and are called bare parameters. m_B is in general not the same as the mass of a particle, because interactions contribute to energy and thus to mass.

The idea of renormalization is that physical quantities are UV finite when written in terms of the particle mass (which is measurable) and some physical measure of the interaction strength like, e.g., a scattering amplitude at a certain energy.

Instead of the physically measurable parameters one may use parameters related to the physical ones by finite equations. Example: $\overline{MS}$ parameters

Such parameters are called renormalized parameters.

Let m, λ be renormalized mass and coupling.

Without interactions: $m_B = m \Rightarrow$

$$m_B^2 = m^2 + \delta m^2 \qquad \text{where } \delta m^2 = O(\lambda)$$

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choose m as the physical particle mass.

We had $\Delta(k) = \int d^dx\, e^{ikx} \langle \varphi(x)\varphi(0)\rangle_0$

and $\Delta(k) = \dfrac{1}{-k^2+m^2}$ without interactions ($m_B = m$)

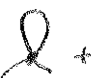
We considered $k^0 = i\omega_n$, but $\Delta(k)$ can be analytically continued. When $\vec{k}=0$, it has poles at

$k^0 = \pm m$

<u>Full 2-point function</u>:

$$G(x) := \frac{1}{Z}\int \mathcal{D}\varphi\, e^{iS}\, \varphi(x)\varphi(0)$$

where $S = S_0 + S_{int}$ is the <u>full action</u>,

$$G(k) = \int d^dk\, e^{ikx}\, G(x) = \frac{1}{-k^2 - m_B^2 - \Pi}\,, \quad \Pi =$$  $+ \dots$

One can show (see QFT) that for $T=0$, $G(k^0,\vec{0})$ has poles at $k^0 = \pm m$, where m is the physical particle mass, also called <u>pole mass</u>.

One finds

$$m_B^2 = m^2 + \underbrace{3\lambda_B \frac{m^2}{(4\pi)^2}\left(\frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{m^2}{4\pi}\right)}_{=\delta m^2} + O(\lambda_B^2)$$

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mass dimension in dimensional regularization:

$$0 = [S] = \left[\int d^{d+1} \mathcal{L}\right] \Rightarrow [\mathcal{L}] = d+1$$

$$[(\partial\varphi)^2] = d+1 \quad \Rightarrow \quad [\varphi^2] = d-1$$

$$[\lambda_B \varphi^4] = [\lambda_B] + \underbrace{2(d-1)}_{=2d-2} \overset{!}{=} d+1 \quad \Rightarrow$$

$$[\lambda_B] = 3 - d$$

Define the renormalized coupling such that it is dimensionless also for $d \neq 3$.

$$\lambda_B = \mu^{3-d} \lambda \left(1 + O(\lambda)\right) = \mu^{2\varepsilon} \lambda \left(1 + O(\lambda)\right)$$

with some mass scale μ.
[μ should not be confused with a chemical potential]

P_0 is a function of m_B and T. It does not depend on λ.

Write

$$(*) \quad P_0 = P_0(m_B^2, T) = P_0(m^2 + \delta m^2, T) = P_0(m^2, T) + \frac{dP_0}{dm^2}\delta m^2 + O(\lambda_B^2)$$

Furthermore, we had

$$\frac{dP_0}{dm^2} = -\frac{1}{2}(I + I_T)$$

When plugged into (*), we obtain the term $(-\frac{1}{2}) I \,\delta m_1^2$ which is T-independent. It is removed by subtracting the T=0 pressure. What remains is

$$P_0(m_B^2, T) = P_0(m^2, T) - \frac{1}{2} I_T \,\delta m_1^2$$

This has to be combined with

$$P_1 = -\frac{3}{4}\lambda_B [\Delta(0)]^2 = -\frac{3}{4}\lambda_B [I + I_T]^2$$

$$= -\frac{3\lambda_B}{4}\{\underbrace{I^2}_{T\text{-independent, drop it}} + 2 I I_T + \underbrace{I_T^2}_{\text{UV finite}}\}$$

We had

$$I = -\frac{m^2}{(4\pi)^2}\left\{\frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{m^2}{4\pi}\right\}$$

and we do not need to distinguish m^2 and m_B^2 because it gets multiplied by λ_B.

P contains

$$-\frac{1}{2} I_T \,\delta m_1^2 - \frac{3\lambda_B}{4} 2 I I_T =$$

$$= \lambda_B I_T \frac{m^2}{(4\pi)^2}\left(\frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{m^2}{4\pi}\right)\left(-\frac{1}{2}\cdot 3 + \frac{3}{4}\cdot 2\right) = 0$$

The $\frac{1}{\varepsilon}$ pole terms cancel, and we can remove the regulator by taking $\varepsilon \to 0$

$$\boxed{P = P_0(m^2, T) - \frac{3}{4}\lambda\, I_T^{\,2} + O(\lambda^2)}$$

High - T expansion:

$$P_0(m^2, T) = T^4 \left\{ \frac{\pi^2}{90} - \frac{m^2}{24T^2} + \frac{m^3}{12\pi T^3} + \cdots \right\}$$

$$I_T = \frac{T^2}{12} - \frac{mT}{4\pi} + O(m^2)$$

$$I_T^{\,2} = \frac{T^4}{(12)^2} - 2\,\frac{mT^3}{12 \cdot 4\pi} + O(m^2 T^2)$$

$$3 I_T^{\,2} = \frac{T^4}{4 \cdot 12} - \frac{mT^3}{4 \cdot 2\pi} + O(m^2 T^2)$$

$$\boxed{\begin{aligned} P = T^4 \Big\{ & \frac{\pi^2}{90} - \frac{m^2}{24T^2} + \frac{m^3}{16\pi T^3} + O(m^4/T^4) \\ & - \frac{\lambda}{4}\left[\frac{1}{48} - \frac{m}{8\pi T} + O(m^2/T^2)\right]\Big\} + O(\lambda^2) \end{aligned}}$$

NB: The $O(\lambda)$ contribution contains a term linear in m.

Expectation: at $O(\lambda^2)$ there will be a $\frac{1}{m}$ contribution.

In the $m \to 0$ limit this would be divergent.

In fact, we will see that in this limit the next term in the perturbative expansion is not $O(\lambda^2)$ but instead $O(\lambda^{3/2})$!