2.4 Interacting field

Now let

$$\mathcal{L}_0 = \frac{1}{2}\partial\varphi^2 - \frac{m^2}{2}\varphi^2 \quad , \quad \mathcal{L}_{int} = -\frac{\lambda}{4}\varphi^4$$

λ : coupling constant

NB: m and λ are bare parameters, in the presence of interaction m will not be the physical particle mass.

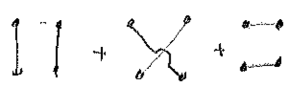
$$\underbrace{P}_{}\ \overset{P \text{ at } \lambda=0}{-\ P_0} = \frac{T}{V}\left[\langle iS_{int}\rangle_0 + \frac{1}{2}\langle (iS_{int})^2\rangle_0 - \frac{1}{2}\left(\langle iS_{int}\rangle_0\right)^2 + O(\lambda^3)\right]$$

$$= P_1 + P_2 + O(\lambda^3)$$

$$P_1 = \frac{T}{V}\ \underbrace{i\int d^4x}_{=\int_0^\beta d\tau \int d^3x}\left(-\frac{\lambda}{4}\right)\langle \varphi^4(x)\rangle_0$$

we had

$$\langle\varphi(x_1)\cdots\varphi(x_4)\rangle_0 \overset{\text{Wick}}{=}$$



$$\left(x_1 \;\; x_2 = \Delta(x_1 - x_2)\right)$$

$$\langle\varphi^4(x)\rangle = 3\left[\Delta(0)\right]^2$$

$$P_1 = -\frac{3\lambda}{4}\left[\Delta(0)\right]^2$$

This is represented by the Feynman diagram

Let us now compute $\Delta(0)$ explicitly.

We had

$$(*) \qquad \Delta(0) = T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k}\left[1 + 2 f_B(E_k)\right]$$

which is UV divergent. To make sense out of this we need to regularize it.
This can be done by

<u>dimensional regularization</u>

Instead of 3, consider d spatial dimensions.
For $d<1$ $(*)$ will be convergent and can be computed. The result may then be analytically continued to other values of d.

Def.
$$\int_{\vec{k}} := \int \frac{d^dk}{(2\pi)^d}$$

Then
$$\Delta(0) = I + I_T$$

$$I = \int_{\vec{k}} \frac{1}{2E_k}, \qquad I_T = \int_{\vec{k}} \frac{f_B(E_k)}{E_k}$$

For $E \gg T$ $\quad f_B(E) \simeq e^{-\beta E}$ exponentially suppressed

$\Rightarrow I_T$ is UV finite for $d=3$.

$$I = \frac{1}{2}\int_{\vec{k}} (\vec{k}^2 + m^2)^{-1/2}$$

Compute (we may also need $A \neq \frac{1}{2}$)

$$\int_{\vec{k}} \frac{1}{(\vec{k}^2+m^2)^A} = \frac{\Omega_d}{(2\pi)^d}\int_0^\infty dk\, k^{d-1}(k^2+m^2)^{-A}$$

Spherical coordinates in d dimensions

Substitute $x = \frac{m^2}{k^2+m^2}$, $k^2+m^2 = \frac{m^2}{x}$, $k^2 = m^2\left(\frac{1}{x}-1\right) = m^2\frac{1-x}{x}$

$k = m(1-x)^{1/2}\, x^{-1/2}$

$$dk = m(1-x)^{-1/2} x^{-3/2}\left[\tfrac{1}{2}(-1)x - \tfrac{1}{2}(1-x)\right]dx = -\frac{m}{2}(1-x)^{-\frac{1}{2}} x^{-3/2}\, dx$$

$$\int_0^\infty dk\, k^{d-1}(k^2+m^2)^{-A} = \tfrac{1}{2} m^{d-2A}\int_0^1 dx\,(1-x)^{-\frac{1}{2}+\frac{d-1}{2}}\, x^{-\frac{3}{2}-\frac{d-1}{2}+A}$$

$$= \frac{m^{d-2A}}{2}\int_0^1 dx\,(1-x)^{d/2-1}\, x^{-d/2-1+A}$$

Beta function   Euler Γ-function

$$\int_0^1 dx\,(1-x)^{\alpha-1} x^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \Rightarrow$$

$$\int_0^\infty dk\, k^{d-1}(k^2+m^2)^{-A} = \frac{m^{d-2A}}{2}\,\frac{\Gamma(d/2)\,\Gamma(A-\frac{d}{2})}{\Gamma(A)}$$

Compute Ω_d:

$$\sqrt{\pi}^{\,d} = \left(\int dx\, e^{-x^2}\right)^d = \int d^dx\, \exp\left(-\sum_{i=1}^d x_i^2\right) = \int d\Omega_d \int_0^\infty dx\, x^{d-1} e^{-x^2}$$

$$\underset{y=x^2}{=} \Omega_d \int_0^\infty dy\, \tfrac{1}{2} y^{-1/2}\, dy\; y^{\frac{d-1}{2}} e^{-y} = \frac{\Omega_d}{2}\int_0^\infty dy\, y^{\frac{d}{2}-1} e^{-y}$$

$$= \frac{\Omega_d}{2}\Gamma(d/2) \Rightarrow \boxed{\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}}$$

$$\frac{\pi^{d/2}}{(2\pi)^d} = \left(\frac{\pi}{4\pi^2}\right)^{d/2} \Rightarrow$$

$$\boxed{\int_{\vec{k}} \frac{1}{(\vec{k}^2+m^2)^A} = \frac{m^{d-2A}}{(4\pi)^{d/2}}\,\frac{\Gamma(A-d/2)}{\Gamma(A)}}$$

Now we need $A = 1/2$

$$\Gamma(A - d/2) = \Gamma(\underbrace{1/2 - d/2}_{\to -1 \text{ for } d \to 3})$$

Γ has poles at $0, -1, -2, \dots$

Def. ε through $d \equiv 3 - 2\varepsilon$

$$\Gamma(1/2 - \tfrac{d}{2}) = \Gamma(-1+\varepsilon)$$

$$\Gamma(1+x) = x\Gamma(x) \Rightarrow \Gamma(1+\varepsilon) = \varepsilon\Gamma(\varepsilon) = \varepsilon(\varepsilon-1)\Gamma(-1+\varepsilon)$$

$$\Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon}\frac{1}{1-\varepsilon}\Gamma(1+\varepsilon)$$

poles at $\varepsilon = 0, 1$, corresponding to $d = 3, 1$.

$$\Gamma(1/2) = \sqrt{\pi}$$

$$I = \frac{1}{2}\int_{\vec{k}} \frac{1}{(\vec{k}^2 + m^2)^{1/2}} = \frac{1}{2}\frac{m^{3-2\varepsilon-1}}{(4\pi)^{3/2-\varepsilon}}\frac{1}{\pi^{1/2}}\left(-\frac{1}{\varepsilon}\right)\frac{1}{1-\varepsilon}\Gamma(1+\varepsilon)$$

$$= \underbrace{-\frac{1}{2}\frac{m^2}{4^{3/2}\pi^2}}_{= -\frac{m^2}{(4\pi)^2}}\left(\frac{m^2}{4\pi}\right)^{-\varepsilon}\frac{1}{\varepsilon}\frac{1}{1-\varepsilon}\Gamma(1+\varepsilon)$$

analytically continue and Laurent expand around $\varepsilon = 0$:

$\Gamma(1+\varepsilon) = 1 - \gamma\varepsilon + \mathcal{O}(\varepsilon^2)$, $\gamma = 0.577$ Euler Mascheroni constant

$$I = -\frac{m^2}{(4\pi)^2}\left\{\frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{m^2}{4\pi}\right\}$$

For the finite T-dependent part I_T we can also write the high- and low T-expansion.

For the order λ^0 contribution to P we had

$$(*) \qquad \frac{dP_0}{dm^2} = -\frac{1}{2}\,\Delta(0) = -\frac{1}{2}\left(I + I_T \right)$$

high T-expansion:

$$P_0(T) - P_0(0) = T^4 \left\{ \frac{\pi^2}{90} - \frac{m^2}{24T^2} + \frac{m^3}{12\pi T^3} + \cdots \right\}$$

$$\Rightarrow \quad I_T = -2T^4 \left\{ -\frac{1}{24T^2} + \frac{3}{2}\,\frac{m}{12\pi T^3} + \cdots \right\}$$

$$= \frac{T^2}{12} - \frac{mT}{4\pi} + \cdots$$

We can also use (*) to compute $P_0(T=0)$:

$$P_0(0) = -\frac{1}{2}\int_0^{m^2} dm'^2\, I(m'^2) = -\frac{1}{2}\int_0^{m^2} dm'^2$$

$$= -\frac{1}{2}\int_0^{m^2} dm'^2 \left(-\frac{1}{(4\pi)^{2-\varepsilon}}\right) \left(m'^2\right)^{1-\varepsilon} \frac{1}{\varepsilon}\,\frac{1}{1-\varepsilon}\,\Gamma(1+\varepsilon)$$

$$= \frac{1}{2(4\pi)^{2-\varepsilon}}\,\frac{1}{\varepsilon}\,\frac{1}{1-\varepsilon}\,\Gamma(1+\varepsilon)\,\frac{1}{2-\varepsilon}\left(m^2\right)^{2-\varepsilon}$$

$$= \frac{1}{4}\,\frac{m^4}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} + \frac{3}{2} - \gamma - \ln\frac{m^2}{4\pi} \right\} + O(\varepsilon)$$