'0/19

2.3 High temperature expansion

For ideal boson gas we had for $s=0, \mu=0$

$$(*)\quad P = -T \int \frac{d^3k}{(2\pi)^3} \ln\left(1 - e^{-\beta E_k}\right)$$

$$E_k = \sqrt{\vec{k}^2 + m^2}$$

For $T^2 \gg m^2$ we expanded

$$P = T^4 \left\{ \frac{\pi^2}{90} - \frac{m^2}{24T^2} + \dots \right\}$$

Naively, the next term in the expansion of $(*)$ would be $O(m^4)$

However, this is wrong: The coefficient of this term would contain an integral like

$$\int_0^\infty dx \frac{1}{(e^x - 1)^2} (\dots) \quad \left(\text{we had } x = \frac{|\vec{k}|}{T}\right)$$

which is divergent at $x=0$. (IR divergence)

This indicates that $(*)$ is not an analytic function of m^2. [this problem does not arise for fermions]

The next term in the high-T expansion of P_{id} turns out to be $O(m^3 T)$.

This can be derived from $(*)$, but we can also obtain it from perturbation theory.

..3

Mass dependence of P

Consider a free field and replace

$$m^2 \to m^2 + \delta m^2$$

This is still a free field theory, which we have already solved. But by treating δm^2 as an "interaction" (or better: perturbation) we still learn something interesting.

$$P - P_0 = \frac{T}{V} \langle i S_{int} \rangle_0 + \mathcal{O}(\delta m^4)$$

where now

$$S_{int} = \int d^4x \left(-\tfrac{1}{2}\right) \delta m^2 \varphi^2$$

$$\langle i S_{int} \rangle_0 = -i \frac{\delta m^2}{2} (-i) \int_0^\beta d\tau \underbrace{\int d^3x}_{\to V} \underbrace{\langle \varphi^2(x) \rangle_0}_{= \Delta(x-x) \equiv \Delta(0)}$$

$$= -\frac{\delta m^2}{2} \beta V \Delta(0) \quad \Rightarrow$$

$$P - P_0 = -\frac{\delta m^2}{2} \Delta(0) \quad \Rightarrow \quad \frac{dP}{dm^2} = -\frac{1}{2} \Delta(0)$$

We had

$$\Delta(-i\tau, \vec{x}) = T \sum_{n \in \mathbb{Z}} e^{-i\omega_n \tau} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \frac{1}{\omega_n^2 + \vec{k}^2 + m^2}$$

.3

lec. 3 Matsubara sums

$f_B(\omega) = \frac{1}{e^{\beta\omega}-1}$ Bose-Einstein distribution is

(*) periodic in imaginary frequency with period $i\omega_n$

$$f_B(\omega + i\omega_n) = f_B(\omega) \quad \text{since } e^{\beta i\omega_n} = e^{i\beta n 2\pi T} = 1$$

near $\omega = 0$: $f_B(\omega) \simeq \frac{T}{\omega} + \mathcal{O}(\omega^0) \Rightarrow f_B$

(**) has poles at $\omega = i\omega_n$ with residue T.

$$\frac{1}{2} + f_B(\omega) = \frac{\frac{1}{2}(e^{\beta\omega}-1)+1}{e^{\beta\omega}-1} = \frac{1}{2}\,\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \frac{1}{2}\,\frac{e^{\beta\omega/2}+e^{-\beta\omega/2}}{e^{\beta\omega/2}-e^{-\beta\omega/2}}$$

$$= \frac{1}{2}\coth(\beta\omega/2)$$

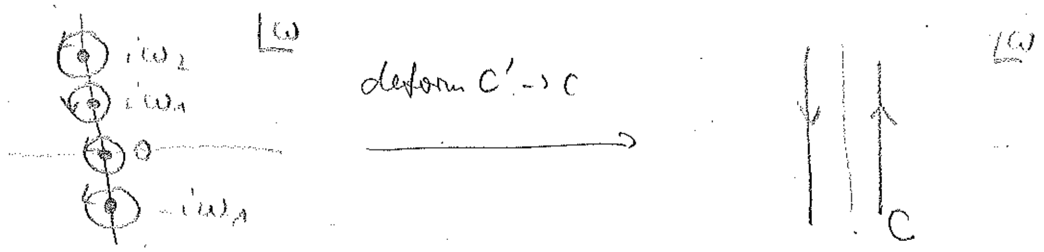
is an odd function of ω and also has the properties (*) & (**).

Let h be analytic near the imaginary axis.

Then

$$T\sum_{n\in\mathbb{Z}} h(i\omega_n) = \int_{C'} \frac{d\omega}{2\pi i}\left[\frac{1}{2} + f_B(\omega)\right] h(\omega)$$

where C' is the following contour:



$$\boxed{T\sum_{n\in\mathbb{Z}} h(i\omega_n) = \int_{C} \frac{d\omega}{2\pi i}\left[\frac{1}{2} + f_B(\omega)\right] h(\omega)}$$

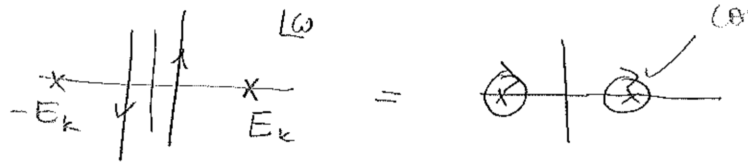
apply this to $h(\omega) = \Delta(\omega, \vec{k})$

$\Delta(\omega,\vec{k}) = \frac{-1}{(\omega - E_k)(\omega + E_k)}$ has poles at $\omega = \pm E_k$, $E_k = \sqrt{\vec{k}^2 + m^2}$

with residues $\mp \frac{1}{2E_k}$

and falls off like $\frac{1}{\omega^2}$ for $\omega \to \infty$.

$$T \sum_{n \in \mathbb{Z}} \Delta(i\omega_n, \vec{k}) =$$



counter-clockwise $\to (-1)$

$$= (-)^2 \frac{1}{2E_k} \left\{ \left[\tfrac{1}{2} + f_B(E_k) \right] - \left[\tfrac{1}{2} + f_B(-E_k) \right] \right\}$$

$$= \frac{1}{2E_k} \left[1 + 2 f_B(E_k) \right]$$

Thus

$$\Delta(0) = T \sum_n \int \frac{d^3k}{(2\pi)^3} \Delta(i\omega_n, \vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left[1 + 2 f_B(E_k) \right]$$

The integral $\int d^3k \frac{1}{2E_k}$ diverges at large $\vec{k}$ (UV divergence)

This divergent contribution is T-independent $\Rightarrow$

$\frac{dP}{dm^2}$ contains a T-independent UV divergence.

We did not see this in chapter 1 because there we put the vacuum energy equal to zero. Since

$$E = \underset{\text{entropy}}{TS} - PV$$

a UV divergent P implies a UV-divergent E.

[This is the famous cosmological constant problem]

We can <u>renormalize</u> P by subtracting the vacuum contribution

$$\frac{dP}{dm^2} = -\int \frac{d^3k}{(2\pi)^3} \frac{f_B(E_k)}{2E_k}$$

compare with (✱):

$$\frac{d}{dm^2}(✱) = -T \int \frac{d^3k}{(2\pi)^3} \underbrace{\frac{dE_k}{dm^2}}_{=\frac{1}{2E_k}} (-\beta) e^{-\beta E_k} (-1) \frac{1}{1-e^{-\beta E_k}} \quad \underline{\text{ok}}$$

Now back to the high T expansion. Before the Matsubara Sum:

$$\Delta(0) = T \sum_{n\in\mathbb{Z}} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + \vec{k}^2 + m^2}$$

If we expand in m^2 here, we see that the IR divergence comes from the $n=0$ term

$$(X) \quad \Big[\Delta(0)\Big]_{\omega_n=0} = T \int \frac{d^3k}{(2\pi)^3} \frac{1}{\vec{k}^2 + m^2}$$

which we treat seperately. It is UV divergent. This divergence is proportional to T and independent of m^2. We already know that the T-dependent part of P is UV finite. Thus the UV divergence in (X) must cancel against contributions from $\omega_n \neq 0$. We can get rid of the UV divergence by differentiating WRT m^2 ⟹

$$\left[\frac{d^2P}{d(m^2)^2}\right]_{\omega_n=0} = -\frac{1}{2}\frac{d}{dm^2}\Delta(0)\Big|_{\omega_n=0} = +\frac{1}{2}T\int \frac{d^3k}{(2\pi)^3}\frac{1}{(\vec{k}^2+m^2)^2}$$

$$= \underbrace{\frac{1}{2}\frac{4\pi}{8\pi^3}}_{=\frac{1}{4\pi^2}} (m^2)^{-1/2} T \underbrace{\int_0^\infty dx\, x^2 (x^2+1)^{-2}}_{=\frac{\pi}{4}} = \frac{T(m^2)^{-1/2}}{16\pi}$$

The resulting contribution to P is

$$\frac{1}{16\pi}\,\frac{2}{3}\cdot 2\,(m^2)^{3/2}\,T = \frac{(m^2)^{3/2}}{12\pi}\,T$$

This is obviously not analytic near $m^2 = 0 \Rightarrow$

high-T expansion for ideal boson gas ($s=0, \mu=0$)

$$\boxed{P = T^4\left\{\frac{\pi^2}{90} - \frac{m^2}{24T^2} + \frac{m^3}{12\pi T^3} + \mathcal{O}(m^4/T^4)\right\}}$$

[NB: the m^4 term also contains $\ln(m^2/T^2)$.]

2.3