2.2 Perturbation theory

Now write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

quadratic in φ, $\partial\varphi$

example: $\mathcal{L}_{int} = -\frac{\lambda}{4}\varphi^4$

$$Z = \int \mathcal{D}\varphi \, e^{i(S_0 + S_{int})}$$ Expand the exponential

$$Z = \int \mathcal{D}\varphi \, e^{iS_0} \left[1 + iS_{int} + \tfrac{1}{2}(iS_{int})^2 + \cdots \right]$$
$$= Z_0 \left[1 + \langle iS_{int} \rangle_0 + \tfrac{1}{2} \langle (iS_{int})^2 \rangle + \cdots \right]$$

with

$$\langle A \rangle_0 := \frac{\int \mathcal{D}\varphi \, e^{iS_0} A}{\int \mathcal{D}\varphi \, e^{iS_0}}$$ "expectation value"

N.B: any temperature dependent factor in the integration measure drop out here

For the pressure P we had $Z = \exp(\beta P V)$, $P = T \ln Z / V$

$$\ln(1+x) = x - \frac{x^2}{2} + \cdots$$
$$\ln Z = \ln Z_0 + \langle iS_{int} \rangle_0 + \tfrac{1}{2} \langle (iS_{int})^2 \rangle_0 - \tfrac{1}{2} (\langle iS_{int} \rangle_0)^2 + \cdots$$

$$P = P_0 + \frac{T}{V} \left[\langle iS_{int} \rangle_0 + \tfrac{1}{2} \langle (iS_{int})^2 \rangle_0 - \tfrac{1}{2} (\langle iS_{int} \rangle_0)^2 + \cdots \right]$$

If we insert an expression for S_{int} we see that we need to compute expectation values like

$$\langle \varphi(x_1)\varphi(x_2) \cdots \varphi(x_n) \rangle_0$$

Wick's theorem: for any even n

$$\boxed{\langle \varphi(x_1)\cdots\varphi(x_n)\rangle_0 = \sum_{\text{contractions}} \langle \varphi(x_1')\varphi(x_2')\rangle_0 \cdots \langle \varphi(x_{n-1}')\varphi(x_n')\rangle_0}$$

$$\{x_1, \dots, x_n\} = \{x_1', \dots, x_n'\}$$

contractions: pairings of $x_1, \dots, x_n$

example: $n = 4$

$$\langle \varphi(x_1)\cdots\varphi(x_4)\rangle_0 = \langle \varphi(x_1)\varphi(x_2)\rangle_0 \langle \varphi(x_3)\varphi(x_4)\rangle_0$$
$$+ \langle \varphi(x_1)\varphi(x_2)\rangle_0 \langle \varphi(x_2)\varphi(x_4)\rangle_0$$
$$+ \langle \varphi(x_1)\varphi(x_4)\rangle_0 \langle \varphi(x_2)\varphi(x_3)\rangle_0$$

graphical interpretation:

N.B: Wick's theorem is a general property of Gauß-integrals

How to compute $\langle \varphi(x_1)\varphi(x_2)\rangle_0$?

For this we will make use of the Fourier representation of φ:

φ periodic "$\varphi(-i\beta, \vec{x}) = \varphi(0, \vec{x})$" $\Rightarrow$

φ can be expanded in a Fourier series

$$\varphi(-i\tau, \vec{x}) = T \sum_{n\in\mathbb{Z}} e^{-i\omega_n \tau} \varphi_n(\vec{x})$$

$\boxed{\omega_n = n\, 2\pi T}$ Matsubara frequencies

$$\varphi_n(\vec{x}) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\, \varphi(-i\tau, \vec{x})$$

We also use the functional derivative defined through

(i) $\frac{\delta}{\delta\varphi(x)}\varphi(x') = \delta(x-x') \quad (= \delta^{(4)}(x-x'))$

(ii) $\frac{\delta}{\delta\varphi(x)}$ satisfies the product rule

Def

$$(*) \qquad Z_0[J] := \frac{\int \mathcal{D}\varphi \exp\left\{ i\left(S_0 + \int d^4x\, J(x)\varphi(x)\right)\right\}}{\int \mathcal{D}\varphi\, e^{iS_0}}$$

generating functional

This is a functional of the field J. Expectation values of φ are obtained by functional differentiation:

$$\langle\varphi(x_1)\,\varphi(x_2)\rangle_0 = \frac{\delta Z_0[J]}{i\delta J(x_1)\, i\delta J(x_2)}\bigg|_{J=0}$$

Now compute $Z_0[J]$. $\int d^4x\, \partial_\mu\varphi\,\partial^\mu\varphi \overset{IBP}{=} -\int \varphi\,\partial^2\varphi \Rightarrow$

$$S_0 + \int d^4x\, J\varphi = \int d^4x \left\{ -\tfrac{1}{2}\varphi(\partial^2+m^2)\varphi + J\varphi \right\}$$

where $\int d^4x \equiv \int_0^{-i\beta} dt \int d^3x$

Complete the square by shifting the integration variable $\varphi \to \varphi + \varphi'$ in $(*)$, where φ' will be determined shortly. Use $\int d^4x\, \varphi'(\partial^2+m^2)\varphi = \int d^4x\, \varphi(\partial^2+m^2)\varphi' \Rightarrow$

$$S_0 + \int d^4x\, J\varphi \to \int d^4x \left\{ -\tfrac{1}{2}\varphi(\partial^2+m^2)\varphi - \varphi(\partial^2+m^2)\varphi' - \tfrac{1}{2}\varphi'(\partial^2+m^2)\varphi' + J\varphi + J\varphi' \right\}$$

Choose φ' such that

$$(**)\qquad (\partial^2 + m^2)\varphi' = J$$

Then the terms linear in φ cancel, and the integral over φ is $\int \mathcal{D}\varphi\, e^{iS_0}$ which cancels in $(*)$.

Solve $(**)$ by FT: [J should also be periodic]

$$\partial^2 = \partial_t^2 - \nabla^2 = -\partial_\tau^2 - \nabla^2 \xrightarrow{FT} \omega_n^2 + \vec{k}^2 = -k^2$$

Write $k^0 = i\omega_n \Rightarrow$

$$\varphi_n'(\vec{k}) = \Delta(k)\, J_n(\vec{k}) \qquad \text{with}$$

$$\Delta(k) := \frac{1}{-k^2 + m^2} = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2}$$

NB: unlike in real time we don't need an $i\varepsilon$-prescription

Inverse FT $\rightarrow$

$$\varphi'(-i\tau, \vec{x}) = T\sum_n \int \frac{d^3k}{(2\pi)^3}\, e^{-i\omega_n\tau} e^{i\vec{k}\vec{x}}\, \Delta(k)\, \underbrace{J_n(\vec{k})}_{= \int_0^\beta d\tau' \int d^3x'\, e^{i\omega_n\tau'} e^{-i\vec{k}\vec{x}'} J(x')}$$

$$= \int_0^\beta d\tau' \int d^3x'\, \Delta(x - x')\, J(x')$$

where

$$\Delta(x - x') = T\sum_n \int \frac{d^3k}{(2\pi)^3}\, e^{-i\omega_n(\tau - \tau')} e^{i\vec{k}(\vec{x} - \vec{x}')}\, \Delta(k)$$

Then

$$Z_0[J] = \exp\Big\{ i\int d^4x\, \underbrace{\Big[-\tfrac{1}{2}\varphi' \overbrace{(\partial^2 + m^2)\varphi'}^{= J} + J\varphi' \Big]}_{= \frac{1}{2} J\varphi'} \Big\} \Rightarrow$$

Write

$$\varphi'(x) = i \int d^4x' \Delta(x-x') J(x')$$ which gives

$$\boxed{Z_0[J] = \exp\left\{ -\frac{1}{2} \int d^4x \int d^4x \; J(x)\, \Delta(x-x')\, J(x') \right\}}$$

more explicitly:

$$\boxed{\begin{aligned} Z_0[J] = \exp\Big\{ \frac{1}{2} \int_0^\beta d\tau\, d\tau' \int d^3x\, d^3x' \; & J(-i\tau, \vec{x})\, \Delta(-i(\tau-\tau'), \vec{x}-\vec{x}') \\ & \cdot J(-i\tau', \vec{x}') \Big\} \end{aligned}}$$

In any case, we obtain

$$\boxed{\langle \varphi(x_1)\, \varphi(x_2) \rangle_0 = \Delta(x_1 - x_2)},$$

One can also use our result to compute n-point functions and to check Wick's theorem.