2 Scalar field theory

2.1 Path integral for Z

in QM: probability amplitude for a particle to go from $\vec{x}_a$ at time t_a to $\vec{x}_b$ at time t_b

position operator, eigenvalue

$$A = \langle \vec{x}_b | U(t_b, t_a) | \vec{x}_a \rangle \quad , \quad \vec{x} | \vec{x}_a \rangle = \vec{x}_a | \vec{x}_a \rangle$$

$$U = T \exp\left\{ i \int_{t_a}^{t_b} dt \, H(t) \right\}$$

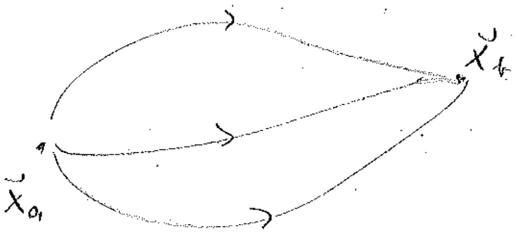
time evolution operator, T: time ordering operator

path integral representation, version #1:

$$A = \int_{\substack{\vec{x}(t_a) = \vec{x}_a \\ \vec{x}(t_b) = \vec{x}_b}} \mathcal{D}\vec{x} \, \mathcal{D}\vec{p} \, \exp\left\{ i \int_{t_a}^{t_b} dt' \left[\dot{\vec{x}} \cdot \vec{p} - H \right] \right\}$$

when H is quadratic in $\vec{p}$, $H = \frac{\vec{p}^2}{2m} + V(\vec{x})$ $\Rightarrow$ version #2

$$A = \int_{\vec{x}(t_a) = \vec{x}_a, \vec{x}(t_b) = \vec{x}_b} \mathcal{D}\vec{x} \, \exp e^{iS[\vec{x}]}$$



$$S[\vec{x}] = \int_{t_a}^{t_b} dt \, L(\dot{\vec{x}}, \vec{x})$$ action

$$L = \tfrac{1}{2} m \dot{\vec{x}}^2 - V$$ Lagrangian

Fields Why fields?

(i) there are fields in nature (electromagnetic, ...)

(ii) all known particles can be understood as quanta of fields

Real scalar field

real scalar field φ, action $S = \int dt\, d^3x\, \mathcal{L} = \int d^4x\, \mathcal{L}$ $(c=1)$

canonical momentum $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$

Hamiltonian density $\mathcal{H} = \pi \dot{\varphi} - \mathcal{L}$

Hamiltonian $H = \int d^3x\, \mathcal{H}$

For $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi\, \partial^\mu \varphi - V(\varphi) = \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} (\nabla \varphi)^2 - V$:

$$\pi = \frac{\partial}{\partial \dot{\varphi}} \left(\frac{1}{2} \dot{\varphi}^2 \right) = \dot{\varphi}$$

$$\mathcal{H} = \pi^2 - \left[\frac{1}{2} \pi^2 - \left(\frac{1}{2} \nabla \varphi\right)^2 - V \right] = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \varphi)^2 + V$$

One can write the Hamiltonian formulation such that it resembles the mechanics version more closely:

$$L = \int d^3x\, \mathcal{L}, \quad \pi(\vec{x}) = \frac{\delta L}{\delta \dot{\varphi}(\vec{x})}, \quad H = -L + \int d^3x\, \pi(\vec{x})\, \dot{\varphi}(\vec{x})$$

$|\varphi_a\rangle$ eigenvector of the field operator

$$\varphi(\vec{x}) |\varphi_a\rangle = \varphi_a(\vec{x}) |\varphi_a\rangle \quad \text{(Schrödinger pic.)}$$

path integral version #1

$$\boxed{\langle\varphi_b|U(t_b,t_a)|\varphi_a\rangle = \int\limits_{\substack{\varphi(t_a,\vec{x})=\varphi_a(\vec{x})\\ \varphi(t_b,\vec{x})=\varphi_b(\vec{x})}} \mathcal{D}\varphi\,\mathcal{D}\pi\, \exp\left\{i\int_{t_a}^{t_b}dt\int d^3x\,\left[\pi\dot{\varphi}-\mathcal{H}\right]\right\}}$$

Integral over all space-time histories $\varphi(t,\vec{x})$

If H is quadratic in π, like

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi) + V(\varphi)$$

one can integrate out π $\Rightarrow$ version #2

$$\boxed{\langle\varphi_b|U(t_b,t_a)|\varphi_a\rangle = \int\limits_{\substack{\varphi(t_a,\vec{x})=\varphi_a(\vec{x})\\ \varphi(t_b,\vec{x})=\varphi_b(\vec{x})}} \mathcal{D}\varphi \quad e^{iS}}$$

remark: for fermion fields one does not integrate out the canonical momenta

partition function

Consider $\mu=0$, H time-independent

$$Z = \operatorname{tr} e^{-\beta H}$$

use field-eigenbasis for the trace,

$$Z = \int\mathcal{D}\varphi_a \ \langle\varphi_a|e^{-\beta H}|\varphi_a\rangle$$

Integral over field configurations $\varphi_a(\vec{x})$

1.2 normalization: $\int\mathcal{D}\varphi_a\ |\varphi_a\rangle\langle\varphi_a| = 1$

$e^{-\beta H}$ can be viewed as a time evolution operator:

$$e^{-\beta H} = e^{-i(-i\beta)H} = U(-i\beta, 0) \Rightarrow$$

path integral #1

$$Z = \int_{\varphi(0,\vec{x}) = \varphi(-i\beta,\vec{x})} \mathcal{D}\varphi\, \mathcal{D}\pi \, \exp\left\{ i \int_0^{-i\beta} dt \int d^3x \, [\pi\dot{\varphi} - \mathcal{H}] \right\}$$

and #2

$$Z = \int_{\varphi(0,\vec{x}) = \varphi(-i\beta,\vec{x})} \mathcal{D}\varphi \, e^{iS}$$

where now $S = \int_0^{-i\beta} dt \int d^3x \, \mathcal{L}$

The field now lives in <u>imaginary time</u>, and φ is <u>periodic</u> in imaginary time.

Write $t = -i\tau$ with $\tau \in \mathbb{R}$,

$$\int_0^{-i\beta} dt = -i \int_0^{\beta} d\tau \quad , \quad \dot{\varphi} = \frac{\partial \varphi}{\partial t} = i \frac{\partial \varphi}{\partial \tau}$$

$$Z = \int_{\varphi \text{ periodic}} \mathcal{D}\varphi \, \exp\left\{ \int_0^{\beta} d\tau \int d^3x \left[-\tfrac{1}{2}(\partial_\tau \varphi)^2 - \tfrac{1}{2}(\nabla\varphi)^2 - V \right] \right\}$$

NB: $(\partial_\tau\varphi)^2$ and $(\nabla\varphi)^2$ appear with the same sign

"Euclidean path integral"

free field

$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$ describes free spin 0 bosons with mass m.

$$Z_0 = \int_{\varphi\text{ periodic}} \mathcal{D}\varphi \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x \left[(\partial_\tau\varphi)^2 + (\nabla\varphi)^2 + m^2\varphi^2 \right]\right\}$$

This is a Gauß integral and can be computed exactly. This is remarkably tedious, one has to be very careful because after integrating out π the measure is T-dependent. We won't do this computation because we have obtained it already in a much simpler way

$$\ln Z = -\sum_{\vec{p}} \ln\left(1-e^{-\beta E_p}\right) = -V\int \frac{d^3p}{(2\pi)^3} \ln\left(1-e^{-\beta E_p}\right)$$