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Partition sum

consider system with conserved charges Q_a, at rest in volume V.
Thermal equilibrium described by temperature T, chemical potentials μ_a (grand canonical ensemble)

$Z = \mathrm{tr}\left(e^{\beta(\mu_a Q_a - H)}\right)$ partition function

$\beta := \frac{1}{T}$ $\quad (k_B = 1)$, $\quad H =$ Hamiltonian

$Z = e^{-\beta\Omega}$, $\quad \Omega = \Omega(T,\mu,V)$ grand canonical potential:

$$\Omega = -PV \qquad (P = \text{pressure}) \quad \text{or}$$

$$Z = e^{\beta P V}$$

1 Ideal gas

1 particle species "charge" = particle number Q

finite box of size L, $V = L^3$, periodic boundary conditions

momenta $\vec{p} = \vec{n}\,\frac{2\pi}{L}$, $\quad \vec{n} \in \mathbb{Z}^3 \quad (\hbar = 1)$

$Z = \prod_{\vec{p}, s_z} Z_{\vec{p}, s_z}$, $\quad Z_{\vec{p}}$ partition function for sub-system with momentum $\vec{p}$ and spin in z-direction s_z

Fermions: two states for each $\vec{p}$ and s_z:

$|0\rangle$ no particle

$|1\rangle$ 1 particle

$$Z_{\vec{p},s_z} = \langle 0| e^{\beta(\mu Q - H)}|0\rangle + \langle 1| e^{\beta(\mu Q - H)}|1\rangle$$

$$= 1 + e^{\beta(\mu - E_p)}$$

$E_p =$ 1-particle energy

relativistic: $E_p = \sqrt{\vec{p}^{\,2} + m^2}$

NB: here we have chosen H such that $H|0\rangle = 0$ (no vacuum energy)

$$\beta P V = \ln Z = \sum_{\vec{p}, s_z} \ln(1 + e^{\beta(\mu - E_p)})$$

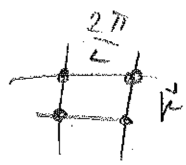
s_z-independence

$$= (2s+1) \sum_{\vec{p}} \ln(1 + e^{\beta(\mu - E_p)})$$

large volume limit $V \to \infty$ (thermodynamic limit)

$$\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{p}} \to \int d^3p,$$

$$\frac{1}{V} \sum_{\vec{p}} \to \int \frac{d^3p}{(2\pi)^3}$$



$$\boxed{P = (2s+1)\, T \int \frac{d^3p}{(2\pi)^3} \ln(1 + e^{\beta(\mu - E_p)})}$$

bosons: $Z_{\vec{p},s_z} = \sum_{n=0}^{\infty} \langle n| e^{\beta(\mu Q - H)}|n\rangle$ n particles

$$= \sum_{n=0}^{\infty} e^{n\beta(\mu - E_p)} = \frac{1}{1 - e^{\beta(\mu - E_p)}}$$

$$\boxed{P = -(2s+1)\, T \int \frac{d^3p}{(2\pi)^3} \ln(1 - e^{\beta(\mu - E_p)})}$$

$$\int \frac{d^3p}{(2\pi)^3} f\left(\frac{|\vec{p}|}{T}\right) = \underbrace{\frac{4\pi}{8\pi^3}}_{=\frac{1}{2\pi^2}} \int_0^\infty dp\, p^2 f\left(\frac{p}{T}\right) = \frac{1}{2\pi^2} T^3 \int_0^\infty dx\, x^2 f(x)$$

consider $\underline{\mu = 0}$,

$$P = \mp (2s+1) \frac{T^4}{2\pi^2} \int_0^\infty dx\, x^2 \ln\left(1 \mp e^{-\sqrt{x^2+y^2}}\right) \quad \begin{matrix}\text{bosons} \\ \text{fermions}\end{matrix}$$

with $y = \frac{m}{T}$

$\sqrt{x}$

low T approximation

$T \ll m$, $y \gg 1$ (non-relativistic limit)

$$\ln(1 \mp e^{-\sqrt{x^2+y^2}}) = \mp e^{-\sqrt{x^2+y^2}} + \Theta(e^{-2y})$$

substitute $w = \sqrt{x^2+y^2}$, $x = \sqrt{w^2-y^2}$

$$dx = \frac{w\, dw}{\sqrt{w^2-y^2}}$$

$$\int_0^\infty dx\, x^2 \ln\left(1 \mp e^{-\sqrt{x^2+y^2}}\right) = \mp \int_y^\infty dw\, w \sqrt{w^2-y^2}\, e^{-w} + \Theta(e^{-2y})$$

$v = w - y$

$$= \mp e^{-y} \int_0^\infty dv\, (v+y) \sqrt{v^2 + 2vy}\, e^{-v} + \Theta(e^{-2y})$$

$$= \mp e^{-y}\, y \sqrt{2y} \int_0^\infty dv\, v^{1/2} \left(1 + \frac{v}{y}\right)\left(1 + \frac{v}{2y}\right)^{1/2} e^{-v} + \Theta(e^{-2y})$$

$$= \mp \sqrt{2}\, y^{3/2} e^{-y}\, \Gamma\left(\tfrac{3}{2}\right) \left[1 + \Theta\left(\tfrac{1}{y}\right) + \Theta(e^{-2y})\right]$$

$$\Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

this expansion yields an asymptotic divergent series.

$$P = (2s+1)\,\frac{T^4}{2\pi^2}\,\sqrt{2}\,\frac{\sqrt{\pi}}{2}\left(\frac{m}{T}\right)^{3/2} e^{-m/T}$$

$$\boxed{P = (2s+1)\,T^4\left(\frac{m}{2\pi T}\right)^{3/2} e^{-m/T}}$$

same for bosons & fermions

<u>high temperature limit</u>

(ultra-relativistic limit?)

$T \gg m \quad ; \; y \ll 1$

expand $\sqrt{x^2+y^2} = x + \dfrac{y^2}{2x} + \mathcal{O}(y^4)$

$$P = \mp(2s+1)\,\frac{T^4}{2\pi^2}\int_0^\infty dx\, x^2 \underbrace{\ln\left(1 \mp e^{-x}\left[1-\frac{y^2}{2x}\right] + \mathcal{O}(y^4)\right)}$$

$$= \ln\left(1 \mp e^{-x} \pm e^{-x}\frac{y^2}{2x}\right) = \ln(1\mp e^{-x}) + \underbrace{\ln\left(1 \pm \frac{e^{-x}}{1\mp e^{-x}}\frac{y^2}{2x}\right)}$$

$$= \pm\frac{1}{e^x \mp 1}\,\frac{y^2}{2x} + \mathcal{O}(y^4)$$

$$= \mp(2s+1)\,\frac{T^4}{2\pi^2}\int_0^\infty dx\, x^2\left\{\ln(1\mp e^{x}) \pm \frac{y^2}{2}\,\frac{1}{e^x\mp 1}\,\frac{1}{x} + \mathcal{O}(y^4)\right\}$$

$$\boxed{P = (2s+1)\,T^4 \begin{cases} \dfrac{\pi^2}{90} - \dfrac{m^2}{24T^2} + \cdots & \text{bosons} \\[2ex] \dfrac{7\pi^2}{720} - \dfrac{m^2}{48T^2} + \cdots & \text{fermions} \end{cases}}$$

The mass decreases P because it leaves less energy for the spatial momenta.

NB: expanding the integrand to $\mathcal{O}(y^4)$ would give an IR divergent integral for bosons. A careful treatment shows that the next term is $\mathcal{O}(m^3)$.

Electroweak Symmetry restauration

H: Higgs field $\langle H \rangle \neq 0$ Breaks EW Symmetry, gives masses to $W^\pm$, Z and fermions:

$$m \sim g \, |\langle H \rangle| \qquad |\langle H \rangle| \sim 170\,\text{GeV}$$

gauge or Yukawa coupling

for $T \gg m_{t_0} \simeq 176\,\text{GeV}$ top quark mass

$$P \simeq c_0 T^4 - c_2 T^2 |\langle H \rangle|^2 , \quad c_0, c_2 > 0$$

regions with smaller $|\langle H \rangle|$ have larger pressure, are favored $\Rightarrow$

$$\langle H \rangle = 0 \quad \text{at high } T$$

At high T the EW symmetry is restored.

A more detailed calculation shows that this happens when $T \gtrsim 130\,\text{GeV}$