Path integral for non-relativistic particle in 1D

Here it is useful to write operators with a hat.

Let $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

$$\langle x_b | e^{-i\hat{H}t} | x_a \rangle = \langle x_b | \underbrace{e^{-i\hat{H}t/N} \cdots e^{-i\hat{H}t/N}}_{N \text{ factors, } N \text{ large}} | x_a \rangle$$

insert $1 = \int dx\, |x\rangle\langle x|$ on the right of each $e^{-i\hat{H}t/N}$

and $1 = \int \frac{dp}{2\pi} |p\rangle\langle p|$ on the left.

Then we obtain N factors of the form

$$\langle x_{i+1} | p_i \rangle \langle p_i | e^{-i\hat{H}t/N} | x_i \rangle$$

$$= \langle x_{i+1} | p_i \rangle \langle p_i | 1 - i\hat{H}t/N + O(N^{-2}) | x_i \rangle$$

$$= \langle x_{i+1} | p_i \rangle \langle p_i | 1 - i\left(\frac{p_i^2}{2m} + V(x_i)\right) | x_i \rangle + O(N^{-2})$$

$$= \langle x_{i+1} | p_i \rangle \langle p_i | x_i \rangle \exp\left(-i H(p_i, x_i) \frac{t}{N}\right) + O(N^{-2})$$

$$= \exp\left(-i \frac{t}{N} \left[-p_i \frac{x_{i+1} - x_i}{t/N} + H(p_i, x_i)\right) + O(N^{-2})$$

Now take $N \to \infty$.

$$\langle x_b | e^{-i\hat{H}t} | x_a \rangle = \lim_{N\to\infty} \int \prod_{i=1}^{N} \frac{dx_i dp_i}{2\pi} \delta(x_1 - x_a)$$

$$\exp\left(-i \sum_{j=1}^{N} \frac{t}{N} \left[-p_j \frac{x_{j+1} - x_j}{t/N} + H(p_j, x_j) \right] \right)$$

where $x_{N+1} = x_b$

define $t_j := (j-1)\frac{t}{N}$, $x(t_j) := x_j$, $p(t_j) := p_j$

For $N \to \infty$:

$$\frac{x_{j+1} - x_j}{t/N} \to \dot{x}(t_j), \quad \sum_{j=1}^{N} \frac{t}{N} \to \int_0^t dt' \implies$$

$$\langle x_b | e^{-i\hat{H}t} | x_a \rangle$$

$$= \int \mathcal{D}x \mathcal{D}p \exp\left(-i \int_0^t dt' \left[p(t')\dot{x}(t') - H(p(t'), x(t')) \right] \right)$$

N.B. This is also valid when H is not quadratic in p