Problem C5.1 In the lecture we assumed the usual formula for integration by parts to show that

$$\int D\overline{\psi}D\psi \frac{\delta e^{iS}}{\delta\overline{\psi}(x)}\overline{\psi}(x') = -\int D\overline{\psi}D\psi e^{iS}\frac{\delta\overline{\psi}(x')}{\delta\overline{\psi}(x)}$$

Check this in light of the fact that we are dealing with Grassmann fields.
Hint: Try to see whether the relation

$$\int d\overline{\psi}\frac{\partial f}{\partial\overline{\psi}}g = -\int d\overline{\psi} f\frac{\partial g}{\partial\overline{\psi}}$$

holds for

$$f = \overline{\psi}\,\overline{\psi}_1\cdots\overline{\psi}_N, \qquad g = \overline{\psi}\psi_1\cdots\psi_M$$

where all ψs and $\overline{\psi}$s are Grassman variables, and N and M are integers.

Problem C5.2 Verify the following properties of the eigenbras and eigenkets of $b^\dagger$ and b for the fermionic harmonic oscillator:

(a)

$$\langle\overline{\psi}_1|\psi_2\rangle = \exp\left(-\frac{1}{2}\overline{\psi}_1\psi_1 - \frac{1}{2}\overline{\psi}_2\psi_2 + \overline{\psi}_1\psi_2\right)$$

(b)

$$\langle\overline{\psi}|\psi\rangle = 1$$

(c)

$$|\psi\rangle\langle\overline{\psi}| = (1-\overline{\psi}\psi)|0\rangle\langle 0| + \overline{\psi}|0\rangle\langle 1| + \psi|1\rangle\langle 0| - \overline{\psi}\psi|1\rangle\langle 1|$$

Problem H5.1 The full scalar field propagator (or 2-point function) in momentum space can be written as

$$G(i\omega_n,\mathbf{k}) = \frac{1}{\omega_n^2 + \mathbf{k}^2 + m_B^2 + \Pi}. \qquad (*)$$

which defines the function $\Pi(i\omega_n,\mathbf{k})$ (sometimes called self-energy).

(a) Consider φ^4-theory in 3+1 dimensions,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_B^2\varphi^2 - \frac{\lambda_B}{4}\varphi^4.$$

In the path integral, expand G to order λ. Also expand $(*)$ to first order in Π. By comparing the two expansions you find an expression for Π valid at order λ.

(b) Compute Π at leading order in the high-temperature expansion (which is of order T^2).

(c) Insert your result for Π into $(*)$. Then $G(0,\mathbf{k})$ has a pole at $\mathbf{k}^2 = -m_{\text{th}}^2$. What is the value of the so-called **thermal mass** m_{th}?

(d) Compute the contribution from the Matsubara zero mode to the ideal gas pressure using the thermal mass instead of the particle mass m, and compare your result with the one from the infrared resummation.

Problem H5.2 Extend the above theory by adding a massless Dirac fermion ψ which couples to the scalar field via a Yukawa interaction with coupling constant g, i.e. add the term

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - g\varphi)\psi$$

to the Lagrangian. Repeat the steps of problem 5.1 but now for the Yukawa interaction instead of the φ^4 interaction. Note that here the leading contribution to Π is order g^2. Hint: For the Matsubara zero mode contribution to the pressure we need $k^0 = 0$, furthermore we need $|\mathbf{k}| \ll T$. Therefore you can put the external momentum k to zero.