Problem C4.1

(a) When we computed the IR-singular piece of P_2, we encountered the integral

$$\int_{\tau_1 \mathbf{x}_1} \int_{\tau_2 \mathbf{x}_2} \Delta(x_1 - x_2)\Delta(x_2 - x_1)$$

with $\int_{\tau \mathbf{x}} \equiv \int_0^\beta d\tau \int d^3x$. Show that it can be written as

$$\beta V T \sum_n \int_{\mathbf{k}} [\Delta(k)]^2$$

where $k^0 = i\omega_n$ with the Matsubara frequencies ω_n, and $\int_{\mathbf{k}} \equiv \int d^3k/(2\pi)^3$.

(b) Try to generalize your result to show that

$$\int_{\tau_1 \mathbf{x}_1} \cdots \int_{\tau_N \mathbf{x}_N} \Delta(x_1 - x_2) \cdots \Delta(x_N - x_1) = \beta V T \sum_n \int_{\mathbf{k}} [\Delta(k)]^N$$

Problem H4.1 Consider a scalar field theory in two spatial dimensions, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4.$$

(a) Start with the free theory at $\mathcal{O}(\lambda^0)$. Compute the pressure for temperatures $T \gg m$. At which order of m^2/T^2 does this high-temperature expansion break down?

(b) Show that the first non-analytic contribution to the pressure at $\mathcal{O}(\lambda^0)$ is given by

$$P_0 = \frac{m^2 T}{8\pi} \ln\left(\frac{m^2}{T^2}\right).$$

Hint: Use the trick that was used in the lecture, i.e. make a small shift in the mass $m^2 \to m^2 + \delta m^2$ and treat the second term as an interaction.

(c) Now consider the pressure at $\mathcal{O}(\lambda)$. Show that in the high-temperature expansion, one already encounters an infrared divergence at this order (in contrast to $\mathcal{O}(\lambda^2)$ in three spatial dimensions), yielding the contribution

$$P_1 = -\frac{3\lambda}{4}\left[\frac{T}{4\pi} \ln\left(\frac{m^2}{T^2}\right)\right]^2.$$