Problem C2.1 The action of a free relativistic point particle with mass m and velocity $\mathbf{v}$ is

$$S = -m \int dt \sqrt{1 - \mathbf{v}^{\,2}}$$

Compute the canonical momentum $\mathbf{p}$ and the Hamiltonian $H(\mathbf{p}, \mathbf{x})$. Is H quadratic in $\mathbf{p}$? How does version 1 of the corresponding path intergral look like? Can one perform the path integral over momentum to obtain version 2 of the path integral?

Problem C2.2 Compute the entropy and the energy of an ideal massless Bose gas using the results of the lecture.

Problem H2.1 A free scalar field in a finite volume V is given by

$$\varphi(-i\tau, \mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2E_{\mathbf{k}} V}} \left(e^{-ikx} a_{\mathbf{k}} + e^{ikx} a_{\mathbf{k}}^{\dagger} \right),$$

with $kx \equiv k^0 x^0 - \mathbf{k} \cdot \mathbf{x}$, $k^0 = E_{\mathbf{k}} = \sqrt{\mathbf{k}^{\,2} + m^2}$, and $x^0 = -i\tau$. The annihilation/creation operators fulfill the commutation relation

$$\left[a_{\mathbf{k}}, a_{\mathbf{q}}^{\dagger}\right] = \delta_{\mathbf{k},\mathbf{q}}.$$

Use this free-field expansion to show that in the infinite-volume limit the two-point correlator is given by

$$\langle \varphi(-i\tau, \mathbf{x}) \varphi(0) \rangle = T \sum_n \int \frac{d^3k}{(2\pi)^3} e^{-i\omega_n \tau + i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2}$$

Remember from the harmonic oscillator with occupation number n that $\langle n | a^{\dagger} a | n \rangle = n$ for annihilation/creation operators satisfying $\left[a, a^{\dagger}\right] = 1$.

Problem H2.2 Again consider the two-point function

$$\langle \varphi(-i\tau, \mathbf{x}) \varphi(0) \rangle$$

but now without using the free-field expression for φ. Insert complete sets of energy-eigenstates $|n\rangle$ with $H|n\rangle = E_n |n\rangle$ to make the τ dependence explicit. Do these sums converge? What happens when τ turns negative?