Problem 1.1 Consider an ideal, non-relativistic gas of bosons with mass m in a finite box of size L. Let μ be the chemical potential associated with the particle number N.

(a) Use the known partition function to show that

$$N = \sum_{\mathbf{k}} \frac{1}{\exp\left(\beta\left[\frac{\mathbf{k}^2}{2m} - \mu\right]\right) - 1}$$

Which values for μ are allowed?

(b) Show that in the thermodynamic limit the number density $n = N/V$ can be written as

$$n = \left(\frac{m}{2\pi\beta}\right)^{3/2} \sum_{j=1}^{\infty} \frac{(e^{\beta\mu})^j}{j^{3/2}}.$$

Assuming a fixed number density, show that upon decreasing the temperature, μ goes to zero already at a finite temperature T_c. What happens when the temperature drops further below T_c?

(c) Go back to a finite volume. Show that the particle number of the zero mode $\mathbf{k} = \mathbf{0}$ diverges as $T \to T_c$. Show that the first non-zero mode, $\mathbf{k} = (2\pi/L, 0, 0)^T$, and with it all higher modes acquire a finite value. What does that imply for the number density when going to an infinite volume again?

Problem 1.2 Let φ be a real scalar field with the action

$$S = \int d^4x \, \mathcal{L}$$

with the Lagrangian (density)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$$

(a) Derive the equation of motion for φ from the principle of least action 'by hand' by (i) replacing $\varphi \to \varphi + \delta\varphi$ with infinitesimal $\delta\varphi = \delta\varphi(x)$ in S and determine the resulting variation $S \to S + \delta S$, and then (ii) demanding $\delta S = 0$ for arbitrary $\delta\varphi(x)$.

(b) Re-derive the equation of motion from the Euler-Lagrange equation

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$