7.7 The anomalous magnetic moment of the electron/muon

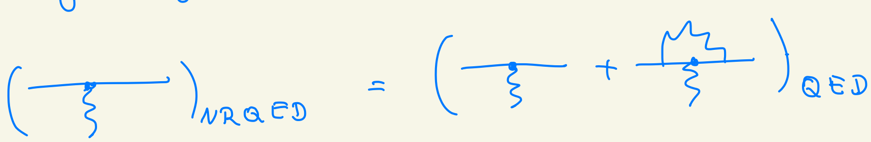
effective Lagrangian for non-relativistic (NR) electrons and soft ($q^\mu \ll m$) photons:

$$\mathcal{L}_{eff} = \chi^+ \left(i D_0 + \frac{\vec{D}^2}{2m} - e g \, \vec{B} \cdot \frac{\vec{\sigma}}{2} \right) \chi$$

non-relativistic QED (NRQED)

Ignoring the effect of hard ($l^\mu \gtrsim m$) virtual particles we found $g = 2$

Now we'll compute the effect of hard virtual particles by computing the amplitude for a NR electron scattering off a soft magnetic field in QED and NRQED and adjust g such that the results match.



To determine g, we only need the spin-dependent part.

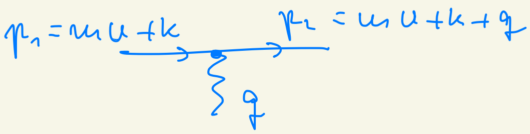
The amplitude in QED

We only need to include the effect of hard loop momenta l.

Since we consider $q^\mu \ll l^\mu, m$ we may put $q^2 = 0$ in $\bar{u}\,\delta\Gamma^\mu u$.

Since $F_1(0) = 0$, the only correction comes from $F_2(0)$.

$u := \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$ $\qquad \vec{q}, \vec{k} = \mathcal{O}(v), \; q^0, k^0 = \mathcal{O}(v^2)$



$= \bar{u}(-ie)u$

$$u(p_1) = \sqrt{2m} \begin{pmatrix} \xi \\ \lambda \end{pmatrix} \qquad \lambda = \mathcal{O}(v)$$

$$\not{p}_1 - m = m(\not{u} - 1) + \not{k} = \begin{pmatrix} 0 & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -2m \end{pmatrix} + \mathcal{O}(v^2)$$

$$(\not{p}_1 - m)u = 0 \Rightarrow \quad \vec{k}\cdot\vec{\sigma} - 2m\lambda = 0,$$

$$u_s(p_1) = \sqrt{2m} \begin{pmatrix} \xi_s \\ \frac{\vec{k}\cdot\vec{\sigma}}{2m}\xi_s \end{pmatrix}$$

Since we consider the scattering off a magnetic field, we need the spatial components:

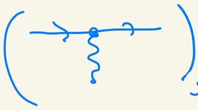
$$\bar{u}_{s_2}(\vec{p}_2)(-ie\gamma^m)\, u_{s_1}(\vec{p}_1) = -ie \; u^\dagger \underbrace{\gamma^0\gamma^m}_{= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^m \\ -\sigma^m & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^m \\ \sigma^m & 0 \end{pmatrix}} u$$

$$= -ie\, 2m \left(\xi^\dagger_{s_2}, \; \xi^\dagger_{s_2} \frac{(\vec{k}+\vec{q})\cdot\vec{\sigma}}{2m} \right) \begin{pmatrix} \sigma^m \frac{\vec{k}\cdot\vec{\sigma}}{2m}\xi_{s_1} \\ \sigma^m \xi_{s_1} \end{pmatrix}$$

$$= -ie \cdot \xi^\dagger_{s_2} \left[\sigma^m \, \vec{k}\cdot\vec{\sigma} + (\vec{k}+\vec{q})\cdot\vec{\sigma}\,\sigma^m \right] \xi_{s_1}$$

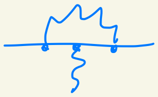
$$= \{\sigma^m, \vec{k}\cdot\vec{\sigma}\} + \vec{q}\cdot\vec{\sigma}\,\sigma^m = 2k^m + \tfrac{1}{2}\{\vec{q}\cdot\vec{\sigma}, \sigma^m\} + \tfrac{1}{2}[\vec{q}\cdot\vec{\sigma}, \sigma^m]$$

$$= \underbrace{2k^m + q^m}_{\text{spin-indep.}} + \underbrace{i\varepsilon^{jmn} q^j \sigma^n}_{\text{spin dependent}}$$



$$\left(\;\cdot\; \right)_{\text{spin dependent}} = e\,\epsilon^{nmj}\, q^n\, \xi^\dagger_{s_2}\sigma^j \xi_{s_1}$$

The contribution from $F_2(0)$:

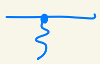
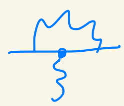


$$= -ie\,\bar{u}\, F_2 \frac{i\sigma^{m\nu} q_\nu}{2m}\, u \simeq \frac{eF_2}{2m}(-1)\, q^n\, \bar{u}\sigma^{mn} u$$

$$\sigma^{mn} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^m \\ -\sigma^m & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^n \\ -\sigma^n & 0 \end{pmatrix} - (m \leftrightarrow n) = \frac{i}{2}\begin{pmatrix} -[\sigma^m,\sigma^n] & 0 \\ 0 & -[\sigma^m,\sigma^n] \end{pmatrix}$$

$$= \epsilon^{mnj}\begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}$$

$$\bar{u}\sigma^{mn} u = \epsilon^{mnj}\, 2m\, \xi^\dagger_{s_2}\sigma^j \xi_{s_1} \quad \Rightarrow$$

 + 

$$\left(\;\cdot\; + \;\cdot\; \right)_{\text{spin dependent}} = e\,\epsilon^{nmj}\, q^n\, \xi^\dagger_{s_2}\sigma^j \xi_{s_1}\left[1 + F_2(0)\right]$$

The 1 in $(1+F_2)$ is reproduced in NRQED by $g=2$ $\Rightarrow$

$$g = 2\left[1 + F_2(0)\right]$$

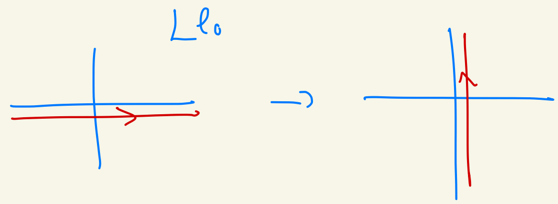
Compute $F_2(0)$

$$F_2(0) = i\, 8 e^2 m^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1-x-y-z)\, z(1-z) \int_\ell \frac{1}{D^3}$$

where now

$$D = \ell^2 - M^2 \quad \text{with} \quad M = (1-z)m$$

Wick rotation :



$$\int_\ell \frac{1}{(\ell^2 - M^2)^3} = i(-1)^3 \int_{\ell_E} \frac{1}{(\ell_E^2 + M^2)^3} \overset{\text{sec. 4.4}}{=} -i \frac{M^{-2}}{(4\pi)^2} \frac{\Gamma(3-2)}{\Gamma(3)}$$

$$= -\frac{i}{(4\pi)^2 M^2} \frac{1}{2 \cdot 1 \cdot \Gamma(1)} = \frac{-i}{(4\pi)^2} \frac{1}{2M^2}$$

$$F_2(0) = i\, 8 e^2 \frac{-i}{2(4\pi)^2} \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1-x-y-z) \frac{z}{1-z}$$

$$= \frac{\alpha}{\pi} \int_0^1 dz \frac{z}{1-z} \int_0^{1-z} dy = \frac{\alpha}{2\pi} \quad \Rightarrow$$

$$\boxed{g - 2 = \frac{\alpha}{\pi}}$$