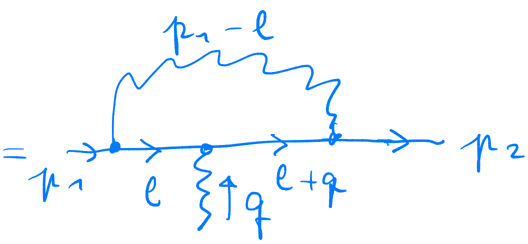
7.6 Vertex correction at order α

$$\bar u \Gamma^\mu u = \bar u (\gamma^\mu + \delta\Gamma^\mu) u$$

Compute $\delta\Gamma^\mu$ at order $\alpha = \frac{e^2}{4\pi}$. def. $\int_\ell \equiv \int \frac{d^d \ell}{(2\pi)^d}$

Use Feynman gauge $\Delta_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu}}{k^2}$

$\bar u(\vec p_2)\, \delta\Gamma^\mu\, u(\vec p_1) =$ 

$p_2 = p_1 + q$

$$= \int_\ell \frac{-i\eta_{\nu\rho}}{(\ell - p_1)^2}\, \bar u(\vec p_2)(-ie\gamma^\nu)\, \frac{i(\not\ell + \not q + m)}{(\ell+q)^2 - m^2}\, \gamma^\mu\, \frac{i(\not\ell + m)}{\ell^2 - m^2} (-ie\gamma^\rho) u(\vec p_1)$$

Combine the 3 denominators using Feynman parameters:

$$\frac{1}{D_1 \cdots D_n} = \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta\Big(1 - \sum_{i=1}^n x_i\Big) \frac{(n-1)!}{[x_1 D_1 + \cdots + x_n D_n]^n}$$

$$\frac{1}{(\ell - p_1)^2}\, \frac{1}{(\ell+q)^2 - m^2}\, \frac{1}{\ell^2 - m^2} = 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1 - x - y - z) \frac{1}{D^3}$$

with $D = x(\ell^2 - m^2) + y([\ell + q]^2 - m^2) + z(\ell - p_1)^2$

$$= \underbrace{\ell^2 + 2\ell\cdot(yq - zp_1)}_{\overset{!}{=} (\ell + yq - zp_1)^2 - \underbrace{(yq - zp_1)^2}_{= -(y^2q^2 - 2yz\, q\cdot p_1 + z^2 p_1^2)}} + y q^2 + z p_1^2 - (x+y) m^2$$

substitute $k = \ell + yq - zp_1$, $\ell = k - yq + zp_1$ $\Rightarrow$

$$D = k^2 - y^2 q^2 + 2yz\, q\cdot p_1 - z^2 \underbrace{p_1^2}_{=m^2} + y q^2 + z p_1^2 - (x+y) m^2$$

$$= k^2 + y(1-y) q^2 + [z(1-z) - (1-z)] m^2 + 2yz\, q\cdot p_1$$

$$= k^2 + y(1-y) q^2 - (1-z)^2 m^2 + 2yz\, q\cdot p_1$$

$p_2 = p_1 + q$, $p_2^2 = m^2 \Rightarrow m^2 = \underbrace{p_1^2}_{=m^2} + 2p_1\cdot q + q^2 \Rightarrow$

$2q\cdot p_1 = -q^2 \Rightarrow$

$$D = k^2 + y(1-y-z) q^2 - (1-z)^2 m^2$$

$$= k^2 + xy\, q^2 - (1-z)^2 m^2$$

which is symmetric under $x \leftrightarrow y$.

$$\bar{u}\, \delta\Gamma^\mu u = -i\, 2e^2 \int_\ell \frac{N^\mu}{D^3}$$

In N one can eliminate the terms containing $\not{q}$ and $\not{p}_1$ by writing $\not{q} = \not{p}_2 - \not{p}_1$ and using $\not{p}_1 u = m u$, $\bar{u}\not{p}_2 = m\bar{u}$ and (excessively) the Dirac algebra.

Def $p := p_1 + p_2$

$$N^\mu = \bar{u}\Big\{ \gamma^\mu \big[2k^2 - 2m^2 + 2m^2 z^2 + 4m^2 z + 2y^2 q^2 - 2y q^2 + 2yz q^2 - 2z q^2 \big] - 4 k^\mu \not{k}$$

$$+ p^\mu \big[-2mz^2 + 2mz \big]$$

$$+ q^\mu \big[4m - 8my + 4myz + 2mz^2 - 6mz \big] \Big\} u$$

$$\int d^4k \, \frac{k^\mu k^\nu}{D^3} = X g^{\mu\nu}, \quad X = \frac{1}{4} \int d^4k \, \frac{k^2}{D^3}$$

$$\int d^4k \, \frac{k^\mu \not{k}}{D^3} = \frac{1}{4} \gamma^\mu \int d^4k \, \frac{k^2}{D^3} \quad \Rightarrow$$

$$N^\mu = \bar{u} \Big\{ \gamma^\mu \big[k^2 + 2m^2 (z^2 + 2z - 1) + 2q^2 \underbrace{(y^2 - y + yz - z)}_{= (y-1)(y+z) = -(1-y)(1-z)}$$

$$+ 2m p^\mu z(1-z) + 2m q^\mu \underbrace{\big[2 - 4y + 2yz + z^2 - 3z \big]}_{= (z-2)(-1+2y+z) = (z-2)(y-x)} \Big\} u$$

D is symmetric under $x \leftrightarrow y$ $\Rightarrow$ the term $2mq^\mu (z-2)(y-x)$ drops out in the Feynman parameter integral. $\Rightarrow$

$$N^\mu \to \bar{u} \Big\{ \gamma^\mu \big[k^2 + 2m^2 (z^2 + 2z - 1) - 2q^2 (1-y)(1-z) \big]$$

$$+ 2m (p_1 + p_2)^\mu z(1-z) \Big\} u$$

Last time we found

$$\bar{u}(p_1+p_2)^\mu u = \bar{u}\left[2m\gamma^\mu - i\sigma^{\mu\nu} q_\nu\right]u$$

This gives

$$N^\mu \to \bar{u}\Big\{\gamma^\mu\Big[\underbrace{4m^2 z(1-z) + 2m^2(z^2+2z-1)}$$

$$= 2m^2\left[2z - 2z^2 + z^2 + 2z - 1\right]$$

$$\overset{!}{=} 2m^2\left[4z - z^2 - 1\right]$$

$$+ k^2 - 2q^2(1-y)(1-z)\Big] - i\,2m\,\sigma^{\mu\nu}q_\nu\, z(1-z)\Big\}u$$

We wrote $\quad \bar{u}\Gamma^\mu u = \bar{u}\left(\gamma^\mu F_1 + i\dfrac{\sigma^{\mu\nu}q_\nu}{2m}F_2\right)u$

Now we found

$$F_1 = 1 + (-i)\,2e^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz\; \delta(1-x-y-z)$$

$$\int_k D^{-3}\left\{k^2 - 2q^2(1-y)(1-z) + 2m^2(4z-z^2-1)\right\} + \mathcal{O}(e^4)$$

$$\sim \int_k k^{-6}\,k^2 \qquad \text{UV - divergent}$$

and

$$(-i)2e^2 \int \cdots (-i)2m\; z(1-z) = \frac{i}{2m}F_2 \quad \Rightarrow$$

$$F_2 = i\,8e^2m^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz\; \delta(1-x-y-z)\, z(1-z)$$

$$\int_k \frac{1}{D^3} \qquad \text{UV finite.}$$