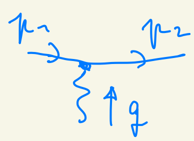
7.5 Higher order: vertex corrections

[Peskin 6.2]

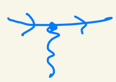
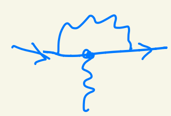
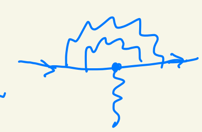
Our S-matrix-element computations were done at leading order. Our non-relativistic approximation ignored the effect of momentum modes with 4-momenta of order m or larger. At higher order these modes can also affect the low momentum ($k^\mu \ll m$) modes through loops.

We will now have a look at higher order corrections to vertex. At leading order

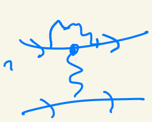


$$= -ie\gamma^\mu$$

Define the "full vertex"

$$-ie\Gamma^\mu(p_1,p_2) :=$$  $+$  $+$  $+ \dots$

Even the 1-loop corrections gets quite involved already.

We will consider $\Gamma^\mu(p_1,p_2)$ for on-shell p_a and sandwiched between $\bar{u}(p_2)$ and $u(p_1)$, as it appears in S-matrix elements like p_1  p_2 .

This can also be used to compute the effect of hard-mode interactions on the magnetic moment of the electron or muon.

Let us first discuss the general structure of

$$\bar{u}(p_2)\,\Gamma^\mu(p_1, p_2)\,u(p_1)$$

The following contribution is already present at tree level ($\equiv$ no loop): $\bar{u}\gamma^\mu u$

At higher orders we could get $p_1^\mu \bar{u}u$ and $p_2^\mu \bar{u}u$ (times numbers and scalars)

What about $\bar{u}\gamma^\mu\gamma^\nu u$? To be a 4-vector like the other terms this would have to be contracted with $p_{1\nu}$ or $p_{2\nu}$. In the first case we would get

$\bar{u}(p_2)\gamma^\mu \not{p}_1 u(p_1)$. Now use $(\not{p} - m)u(p) = 0 \Rightarrow$

$$\bar{u}(p_2)\gamma^\mu \not{p}_1 u(p_1) = m\,\bar{u}(p_2)\gamma^\mu u(p_1)$$

In the second case we would also use the Dirac algebra:

$$\bar{u}(p_2)\underbrace{\gamma^\mu \not{p}_2}_{= \{\gamma^\mu, \not{p}_2\} - \not{p}_2\gamma^\mu = \{\gamma^\mu, \gamma^\nu\}p_{2\nu} - \not{p}_2\gamma^\mu} u(p_1) = \bar{u}(p_2)\left[2p_2^\mu - m\gamma^\mu\right]u(p_1)$$

Similarly, terms with more Dirac matrices give no new structures either $\Rightarrow$

$$(*)\quad \bar{u}\,\Gamma^\mu u = \bar{u}\left[\gamma^\mu a + (p_1+p_2)^\mu b + (p_1 - p_2)^\mu c\right]u$$

where a, b and c are scalars. Since $p_a^2 = m^2$, the only non-trivial scalar is q^2 with

$$q = p_2 - p_1$$

Thus a, b, c only depend on m^2 and q^2.

$\bar{u}\Gamma^\mu u$ can be viewed as a matrix element of the electric current J^μ. E.g., for our computation of g it is due to the interaction of hard (= high momentum) fermions with soft (low momentum) A_μ:

$$\mathcal{L}_{int} = - A^\mu_{soft} J_\mu$$

$$\bar{u}\Gamma^\mu u \sim \int d^4x\, e^{-iqx} \langle \cdots J^\mu(x) \cdots \rangle$$

where q is soft. Current conservation $\partial_\mu J^\mu = 0 \Rightarrow$

$$q_\mu \bar{u}\Gamma^\mu u = 0$$

$q_\mu (*)$, $q = p_2 - p_1$

$$0 = \bar{u}\left[\underset{0}{(\not{p}_2 - \not{p}_1)} a + \underbrace{(p_2^2 - p_1^2)}_{= m^2 - m^2 = 0} b - q^2 c \right] u \Rightarrow$$

$$c = 0$$

The remaining terms can be rewritten using

$$\bar{u}(p_2)\gamma^\mu u(p_1) = \frac{1}{2m}\bar{u}(p_2)\left[(p_1+p_2)^\mu + i\sigma^{\mu\nu}q_\nu\right]u(p_1)$$

with

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

check: $\bar{u}\, i\sigma^{\mu\nu}q_\nu u = -\frac{1}{2}\bar{u}(\gamma^\mu \not{q} - \not{q}\gamma^\mu)u$

$$= -\frac{1}{2}\bar{u}\left(\gamma^\mu(\not{p}_2-\not{p}_1) - (\not{p}_2-\not{p}_1)\gamma^\mu\right)u$$

$$= -\frac{1}{2}\bar{u}\left(2p_2^\mu - m\gamma^\mu - m\gamma^\mu - m\gamma^\mu + 2p_1^\mu - m\gamma^\mu\right)u$$

$$= -\frac{1}{2}\bar{u}\left(2(p_1+p_2)^\mu - 4m\gamma^\mu\right)u$$

$$= \bar{u}\left(-(p_1+p_2)^\mu + 2m\gamma^\mu\right)u \qquad \blacksquare$$

So we can write

$$(*) \qquad \boxed{\bar{u}\Gamma^\mu u = \bar{u}\left(\gamma^\mu F_1 + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2\right)u}$$

with $u = u(\vec{p}_1)$, $\bar{u} = \bar{u}(\vec{p}_2)$, $F_i = F_i(q^2)$ form factors

We will see that F_1 is UV-divergent. It determines the value of the electric charge if it is measured by Coulomb scattering at small momentum transfer $q\to 0$. Then the second term in $(*)$ does not contribute, and

$$F_1(0) = 0$$