7.4 Non-relativistic limit

Consider system on non-relativistic electrons (mass m)
photons with energies $\ll m$

We should recover non-relativistic QM plus relativistic corrections

Let $v \ll 1$ be a typical velocity of the electrons.

idea: expand in v.

estimates: momenta $\sim mv$
kinetic energies $\sim mv^2$

So far we have used the Weyl rep. of γ-matrices
Here it is more convenient to use Dirac rep

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

which is related to Weyl rep. by the unitary transformation

$$\gamma^\mu_{Dirac} = U \gamma^\mu_{Weyl} U^\dagger \quad \text{with} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}$$

free Dirac field: $\psi = \sum_s \int \frac{d^3p}{(2\pi)^3 2p^0} \left\{ e^{-ipx} u_s(\vec{p}) a_s(\vec{p}) + e^{ipx} v_s(\vec{p}) b_s^\dagger(\vec{p}) \right\}$

$(\not{p} - m) u = 0 \qquad \not{p} = m\gamma^0 + O(v)$

$$(\not{p} - m) = m\begin{pmatrix} 0 & 0 \\ 0 & -2\mathbb{1} \end{pmatrix} \Rightarrow u = \begin{pmatrix} \xi \\ 0 \end{pmatrix} + O(v)$$

lower components are small

$p \cdot x = p^0 t - \vec{p} \cdot \vec{x}$

$$p^0 t = \sqrt{m^2 + \vec{p}^2}\, t = m\left(1 + \frac{\vec{p}^2}{2m} + O(v^4)\right) t$$

estimate $\vec{x} \sim v t \Rightarrow \vec{p}\cdot\vec{x} \sim m t v^2$

expectation: the part of ψ which relevant here is obeys

$$\psi(x) = e^{-imt} \cdot (\text{slowly varying terms})$$

slowly means $\sim m v^2$

Ansatz:

$$\psi(x) = e^{-imt} \begin{pmatrix} \chi(x) \\ \eta(x) \end{pmatrix}$$

expectation: $|\chi| \gg |\eta|$

Insert this into

$$\mathcal{L}_e = \bar{\psi}(i\not{D} - m)\psi$$

$$i D_0 \psi = e^{-imt} (m + i D_0) \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

$$\mathcal{L}_e = (\chi^+, \eta^+) \gamma^0 (\gamma^0 m + i\gamma^0 D_0 + i \vec{\gamma}\cdot\vec{D} - m) \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

$\vec{\gamma}\cdot\vec{D} = \gamma^m D_m$, $D_m = \partial_m + ie A^m$

$$\gamma^0 \vec{\gamma}\cdot\vec{D} = \gamma^0 \gamma^m D_m = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \begin{pmatrix} & \sigma^m \\ -\sigma^m & \end{pmatrix} D_m$$

$$= \begin{pmatrix} 0 & \sigma^m \\ \sigma^m & 0 \end{pmatrix} D_m = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{D} \\ \vec{\sigma}\cdot\vec{D} & 0 \end{pmatrix}$$

$$\mathcal{L}_e = (\chi^+, \eta^+) \underbrace{\begin{pmatrix} iD_0 & i\vec{\sigma}\cdot\vec{D} \\ i\vec{\sigma}\cdot\vec{D} & 2m + iD_0 \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix}}_{\stackrel{!}{=} \begin{pmatrix} iD_0 \chi + i\vec{\sigma}\cdot\vec{D}\eta \\ i\vec{\sigma}\vec{D}\chi + (2m + iD_0)\eta \end{pmatrix}}$$

$$= \chi^+ iD_0 \chi + \chi^+ i\vec{\sigma}\cdot\vec{D}\eta + \eta^+ i\vec{\sigma}\cdot\vec{D}\chi + \eta^+(2m + iD_0)\eta$$

χ "light", η "heavy"

we can integrate out η^+, η in the path integral. Since the integral is Gaussian, this is equivalent to solving the eq. of motion.

(x) $$i\vec{\sigma}\cdot\vec{D}\chi + (2m + iD_0)\eta = 0$$

(formal) solution:

$$\eta = \frac{1}{2m + iD_0}(-i)\,\vec{\sigma}\cdot\vec{D}\chi$$

assume that A_μ is sufficiently small such that $D_\mu \sim \partial_\mu$

Then $D_0 \sim mv^2$, $\vec{D} \sim mv$ and $\eta \sim v\chi$ as expected.

Insert (x) in $\mathcal{L}_e$

$$\mathcal{L}_e = \chi^+ iD_0 \chi + \chi^+ i\vec{\sigma}\cdot\vec{D}\eta \quad \Leftrightarrow$$

(*) $$\mathcal{L}_e = \chi^+ \left[iD_0 + \vec{\sigma}\cdot\vec{D}\frac{1}{2m + iD_0}\vec{\sigma}\cdot\vec{D} \right]\chi$$

So far: no approximations

non-relativistic limit

now we can expand in iD_0/m. At leading order

$$\frac{1}{2m + iD_0} = \frac{1}{2m}$$

$$\mathcal{L}_e = \chi^+ \left[iD_0 + \frac{(\vec{\sigma}\,\vec{D})^2}{2m} \right] \chi$$

$$\sigma^i\sigma^j = \frac{1}{2}\{\sigma^i, \sigma^j\} + \frac{1}{2}[\sigma^i, \sigma^j] = \delta^{ij} + i\,\varepsilon^{ijk}\sigma^k$$

$$(\vec{\sigma}\cdot\vec{D})^2 = \sigma^i\sigma^j D_i D_j = \vec{D}^2 + i\,\varepsilon^{ijk}\sigma^k D_i D_j$$

$$= \vec{D}^2 + \frac{i}{2}\sigma^k \varepsilon^{kij} [D_i, D_j]$$

$$[D_i, D_j] = (\partial_i - ieA_i)(\partial_j - ieA_j) - (i \leftrightarrow j)$$

$$= -ie\,(\partial_i A_j - \partial_j A_i) = +ie\,(\partial_i A^j - \partial_j A^i)$$

$$(\vec{\sigma}\cdot\vec{D})^2 = \vec{D}^2 + i\sigma^k\varepsilon^{kij}\, ie\,\partial_i A^j = \vec{D}^2 - e\,\vec{\sigma}\cdot\nabla\times\vec{A}$$

$$= \vec{D}^2 - e\,\vec{\sigma}\cdot\vec{B}$$

$$\boxed{\mathcal{L}_e = \chi^+ \left[iD_0 + \frac{1}{2m}\left(\vec{D}^2 - g\,e\,\vec{B}\cdot\frac{\vec{\sigma}}{2} \right) \right] \chi}$$

with

$$\boxed{g = 2}$$

eq. of motion

$$\left[i D_0 + \frac{1}{2m} \left(\vec{D}^2 - g e \, \vec{B} \cdot \frac{\vec{\sigma}}{2} \right) \right] \chi = 0$$

$iD_0 = i\partial_t + e A_0 \Rightarrow$

$$\boxed{ i \partial_t \chi = \left[- \frac{1}{2m} (\nabla + i e \vec{A})^2 - e A_0 + g \frac{e}{2m} \vec{B} \cdot \frac{\vec{\sigma}}{2} \right] \chi }$$

Pauli equation

describes non-relativistic particle with charge $(-e)$

$$H = \frac{1}{2m} \left(\vec{p} + e\vec{A} \right)^2 - e A_0 + g \frac{e}{2m} \vec{B} \cdot \vec{S}$$

with $\vec{S} = \frac{\vec{\sigma}}{2}$.

energy of magnetic moment $\vec{\mu}$: $H_{int} = -\vec{\mu}\vec{B}$

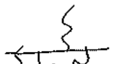
for orbital motion of a particle with charge q:

$$\vec{\mu} = \frac{1}{2} q \, \vec{r} \times \vec{v} = + \frac{1}{2} \frac{q}{m} \vec{r} \times \vec{p} = \frac{q}{2m} \vec{L}$$

for the orbital motion one would find $g = 1$

Our result $g = 2$ for the spin is quite remarkable, and can't be understood classically!

It is not exact, there are QED corrections:

 +  (with $\vec{p} \gtrsim m$) + higer orders.

Expanding (*) to next order gives relativistic corrections: Darwin term, spin-orbit coupling