7.3 Muon pair creation

[Peskin 5.1]

Consider the process

$$e^+ e^- \to \mu^+ \mu^-$$

It requires a center-of-mass energy $\sqrt{s} \geq 2 m_\mu$.
($m_\mu = 106\,\text{MeV}$, $m_e = 511\,\text{keV}$)

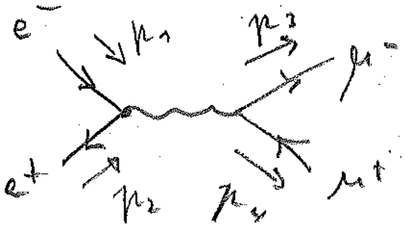
We now have two types ("flavors") of fermions, which we can describe by Dirac spinors ψ_e, ψ_μ.

$$\mathcal{L} = \sum_{f=e,\mu} \bar{\psi}_f (i\not{\partial} + e\not{A} - m_f)\psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

N.B. We use that e and μ interact with exactly the same coupling constant e, with

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

At $\mathcal{O}(e^2)$ there is only 1 Feynman diagram:



S-matrix element:

$$S_{fi} = i\mathcal{M}(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)$$

$$i\mathcal{M} = (ie)^2\, \bar{v}_{e s_2}(\vec{p}_2)\,\gamma_\mu\, u_{e s_1}(\vec{p}_1)\, \Delta_F^{\mu\nu}(p_1+p_2)$$
$$\bar{u}_{\mu s_3}(\vec{p}_3)\,\gamma_\nu\, v_{\mu s_4}(\vec{p}_4)$$

First check whether $\mathcal{M}$ is independent of the gauge fixing parameter ξ:

$$\Delta_F^{\mu\nu}(p) = \frac{i}{p^2}\left(-\eta^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

The piece proportional to $(1-\xi)$ contains

$$\bar{v}_{e s_2}(\vec{p}_2)\underbrace{(\not{p}_1 + \not{p}_2)}_{=(\not{p}_1 - m_e)+(\not{p}_2 + m_e)} u_{e s_1}(\vec{p}_1) = 0$$

because $(\not{p}_1 - m_e)\, u_{e s_1}(\vec{p}_1) = (\not{p}_2 + m_e)\, v_{e s_2}(\vec{p}_2) = 0$

$\Rightarrow$ $\mathcal{M}$ is ξ-independent ■

shorthand notation: $u_1 := u_{e s_1}(\vec{p}_1), \ldots$

$$i\mathcal{M} = -e^2\left(\bar{v}_2 \gamma_\mu u_1\right)\left(-\frac{i}{s}\right)\left(\bar{u}_3 \gamma^\mu v_4\right)$$

For the cross section we need

$$|\mathcal{M}|^2 = \frac{e^4}{s^2}\left(\bar{v}_2 \gamma_\mu u_1\right)^* \left(v_2 \gamma_\nu u_1\right)\left(\bar{u}_3 \gamma^\mu v_4\right)^* \left(\bar{u}_3 \gamma^\nu v_4\right)$$

$$(\bar{v}_2 \gamma_\mu u_1)^* = u_1^+ \underbrace{\gamma_\mu^+}_{=\gamma^0\gamma_\mu\gamma^0} \bar{v}_2^+ = \bar{u}_1 \gamma_\mu v_2$$

$$|\mathcal{M}|^2 = \frac{e^4}{s^2} \left(\bar{u}_1 \gamma_\mu \overbrace{v_2 \bar{v}_2}^{\text{dyadic product}} \gamma_\nu u_1\right)\left(\bar{v}_4 \gamma^\mu u_3 \bar{u}_3 \gamma^\nu v_4\right)$$

$$= \frac{e^4}{s^2} \operatorname{tr}(u_1 \bar{u}_1 \gamma_\mu v_2 \bar{v}_2 \gamma_\nu) \operatorname{tr}(v_4 \bar{v}_4 \gamma^\mu u_3 \bar{u}_3 \gamma^\nu)$$

Unpolarized cross section

If the e^+, e^- are unpolarized, and if the spins of μ^+, μ^- are not measured, one needs $|\mathcal{M}|^2$ averaged over in-state spin and summed over the out-state spins:

$$\langle |\mathcal{M}|^2 \rangle := \frac{1}{4} \sum_{s_1, \dots, s_4} |\mathcal{M}|^2$$

Now we can use

$$\sum_s u_s \bar{u}_s = (\not{p} + m), \quad \sum_s v_s \bar{v}_s = (\not{p} - m)$$

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4 s^2} \operatorname{tr}\left[(\not{p}_1 + m_e)\gamma_\mu (\not{p}_2 - m_e) \gamma_\nu\right]$$

$$\operatorname{tr}\left[(\not{p}_4 - m_\mu)\gamma^\mu (\not{p}_3 + m_\mu) \gamma^\nu\right]$$

Since $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$, the trace of the product of an odd number of γ matrices is zero. $\Rightarrow$

$$\mathrm{tr}\left[(\not p_1 + m_e)\gamma_\mu(\not p_2 - m_e)\gamma_\nu\right] =$$

$$= \mathrm{tr}\left[\not p_1 \gamma_\mu \not p_2 \gamma_\nu\right] - m_e^2 \underbrace{\mathrm{tr}\left[\gamma_\mu \gamma_\nu\right]}_{=4\eta_{\mu\nu}}$$

$$\mathrm{tr}\left[\not p_1 \gamma_\mu \not p_2 \gamma_\nu\right] = p_1^\rho p_2^\sigma \underbrace{\mathrm{tr}(\gamma_\rho \gamma_\mu \gamma_\sigma \gamma_\nu)}_{=4(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\rho\sigma}\eta_{\mu\nu} + \eta_{\rho\nu}\eta_{\mu\sigma})}$$

$$= 4(p_{1\mu} p_{2\nu} - \eta_{\mu\nu}\, p_1\cdot p_2 + p_{1\nu} p_{2\mu})$$

$$= 4\left\{p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - \eta_{\mu\nu}(p_1\cdot p_2 + m_e^2)\right\}$$

Neglect $m_e \ll m_\mu$:

$$\langle |\mathcal{M}|^2 \rangle = 4\frac{e^4}{s^2}\left[p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - \eta_{\mu\nu}\, p_1\cdot p_2\right]$$

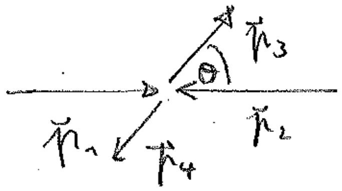
$$\left[p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - \eta^{\mu\nu}(p_3\cdot p_4 + m_\mu^2)\right]$$

$$= \frac{4e^4}{s^2}\Big\{2\, p_1\cdot p_3\, p_2\cdot p_4 + 2\, p_1\cdot p_4\, p_2\cdot p_3$$

$$-2\, p_1\cdot p_2(p_3\cdot p_4 + m_\mu^2) - 2\, p_1\cdot p_2\, p_3\cdot p_4 + 4\, p_1\cdot p_2(p_3\cdot p_4 + m_\mu^2)\Big\}$$

$$= \frac{4e^4}{s^2}\, 2\left\{p_1\cdot p_3\, p_2\cdot p_4 + p_1\cdot p_4\, p_2\cdot p_3 + p_1\cdot p_2\, m_\mu^2\right\}$$

in the center-of-mass system:



$$\vec{p}_1 + \vec{p}_2 = 0 \quad , \quad E_i = \frac{\sqrt{s}}{2}$$

$$|\vec{p}_1| = |\vec{p}_2| = \frac{\sqrt{s}}{2} \quad , \quad |\vec{p}_3| = |\vec{p}_4| = \sqrt{E_3^2 - m_\mu^2} = \sqrt{s/4 - m_\mu^2}$$

$$p_1 \cdot p_2 = \frac{1}{2}(p_1 + p_2)^2 = \frac{s}{2}$$

$$p_1 \cdot p_3 = \frac{s}{4} - \vec{p}_1 \cdot \vec{p}_3 = \frac{s}{4} - \frac{\sqrt{s}}{2} |\vec{p}_3| \cos\theta$$

$$p_1 \cdot p_4 = \frac{s}{4} + \frac{\sqrt{s}}{2} |\vec{p}_3| \cos\theta$$

$$p_2 \cdot p_4 = -\frac{1}{2}\left[(p_2 - p_4)^2 - m_\mu^2\right] = p_1 \cdot p_3$$

$$\langle |\mathcal{M}|^2 \rangle = 8 \frac{e^4}{s^2} \left\{ \left(\frac{s}{4} - \frac{\sqrt{s}}{2} |\vec{p}_3| \cos\theta\right)^2 + \left(\frac{s}{4} + \frac{\sqrt{s}}{2} |\vec{p}_3| \cos\theta\right)^2 + \frac{s}{2} m_\mu^2 \right\}$$

$$= 8 \frac{e^4}{s^2} \left\{ \frac{s^2}{8} + \frac{s}{2} \underbrace{\vec{p}_3^{\,2}}_{= \frac{s}{4} - m_\mu^2} \cos^2\theta + \frac{s}{2} m_\mu^2 \right\} = e^4 \left\{ 1 + 4 \frac{m_\mu^2}{s} + \left(1 - \frac{4 m_\mu^2}{s}\right) \cos^2\theta \right\}$$

differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_3|}{|\vec{p}_1|} \langle |\mathcal{M}|^2 \rangle = \frac{1}{64\pi^2 s} \sqrt{1 - 4 m_\mu^2 / s} \, \langle |\mathcal{M}|^2 \rangle$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4 s} \sqrt{1 - 4 m_\mu^2 / s} \left[1 + 4 \frac{m_\mu^2}{s} + \left(1 - 4 \frac{m_\mu^2}{s}\right) \cos^2\theta \right]}$$

total cross section:

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega} = 2\pi \int_{-1}^{1} d\cos\theta \, \frac{d\sigma}{d\Omega}$$

$$= \frac{\pi \alpha^2}{2s} \sqrt{1 - 4 m_\mu^2/s} \, \underbrace{\left[2\left(1 + 4\frac{m_\mu^2}{s}\right) + \frac{2}{3}\left(1 - 4\frac{m_\mu^2}{s}\right)\right]}_{= \frac{8}{3} + \frac{16}{3}\frac{m_\mu^2}{s} = \frac{8}{3}\left(1 + 2\frac{m_\mu^2}{s}\right)}$$

$$\boxed{\sigma = \frac{4\pi}{3} \frac{\alpha^2}{s} \sqrt{1 - 4 m_\mu^2/s} \left(1 + 2\frac{m_\mu^2}{s}\right)}$$

high energy limit $s \gg m_\mu^2$:

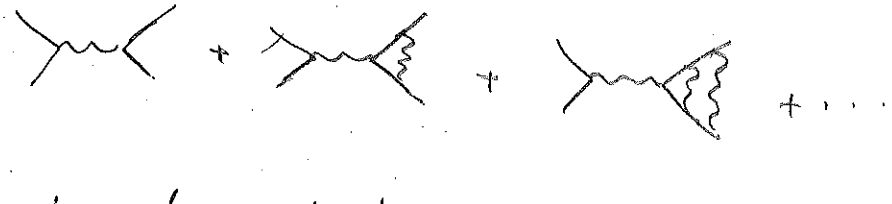
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right) \quad , \quad \sigma = \frac{4\pi}{3}\frac{\alpha^2}{s} \quad \text{only scale} = s$$

near threshold: $\sqrt{s} \to 2 m_\mu$ $\quad \sigma \to 0$

not quite correct: $\mu^+\mu^-$ bound states have $E < 2 m_\mu$

perturbation theory breaks down for $\sqrt{s} \simeq 2 m_\mu$

resummation of multiple photon exchange:



gives bound states.