7 Quantum electrodynamics (QED)

7.1 Gauge invariance

We have seen that gauge invariance ensures that the electromagnetic field has only 2 "degrees of freedom". These correspond to 2 photon polarizations.

To preserve this property, interactions must also be gauge invariant. We want to describe interactions of photons with electrons and positrons.

The free Dirac action is invariant under

(*) $$\psi(x) \mapsto e^{-ie\chi}\,\psi(x)$$

with constant χ arde. It is not invariant when χ depends on x, because

$$\partial_\mu \psi \to e^{-ie\chi}\left[\partial_\mu \psi - ie\,\partial_\mu \chi\right]$$

if

$$A_\mu(x) \to A_\mu - \partial_\mu \chi,$$

then the <u>covariant derivative</u>

$$\boxed{D_\mu := \partial_\mu - ie\,A_\mu}$$

of ψ transform like

$$D_\mu \psi \to e^{-ie\chi}\left[\partial_\mu \psi - ie\,\partial_\mu \chi - ie\left(A_\mu - \partial_\mu \chi\right)\right]$$
$$= e^{-ie\chi} D_\mu \psi \qquad \Rightarrow$$

$\bar\psi D_\mu \psi$ is gauge invariant.

For free Dirac fermions: $\mathcal{L} = \bar\psi(i\not\partial - m)\psi$

replace $\partial\!\!\!/ \to D\!\!\!\!/$, include $\mathcal{L}$ for A_μ $\to$

Lagrangian of QED:

$$\boxed{\mathcal{L} = \bar{\Psi}\,(i\partial\!\!\!/ + e A\!\!\!/ - m)\,\Psi \; - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

$$\underset{\substack{\uparrow \\ \text{free}}}{\mathcal{L} = \mathcal{L}_0} + \mathcal{L}_{int}, \qquad \mathcal{L}_{int} = - A_\mu J^\mu \qquad \text{where}$$

$$\boxed{J^\mu = -e\,\bar{\Psi}\,\gamma^\mu\,\Psi}$$

Note that J^μ is the conserved current corresponding to the $U(1)$ symmetry $(*)$.

The signs are choosen such that the charge $Q = \int d^3x\, J^0$ satisfies

$$[Q, \Psi] = +e\,\Psi$$

which means that in

$$\Psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2p^0} \left\{ e^{-ipx} u_s(\vec{p})\, a_s(\vec{p}) + e^{ipx} v_s(\vec{p})\, b_s^\dagger(\vec{p}) \right\}$$

$b^\dagger$ creates particles with $Q = +e$. Choose $e > 0$, so that these are positrons.

7.2 Perturbation theory

n-point function

$$\langle 0 | T\, \psi(x_1)\, \psi(x_2) \cdots \bar{\psi}(x_1') \cdots A_\mu(x_1'') \cdots | 0 \rangle$$

$$= \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}A\, e^{iS_{eff}}\, \psi(x_1)\, \psi(x_2) \cdots \bar{\psi}(x_1') \cdots A_\mu(x_1'') \cdots$$

with $S_{eff} = S + S_{gf}$,

$$S_{gf} = -\frac{1}{2\xi} \int d^4x\, (\partial \cdot A)^2$$

$$S = S_0 + S_{int}, \qquad S_{int} = -\int d^4x\, J \cdot A$$

expand $e^{iS_{int}} = 1 + iS_{int} + \cdots$,

then apply Wick's theorem $\to$ expansion in e

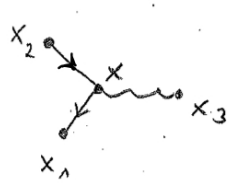
example: <u>3-point function</u> at order e:

$$\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}A\, \psi_\alpha(x_1)\, \bar{\psi}_\beta(x_2)\, A^\mu(x_3)\, e^{iS}$$

$$= \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}A\, e^{iS_0} \int d^4x\, \psi_\alpha(x_1)\, \bar{\psi}_\beta(x_2)\, ie\, \bar{\psi}_\gamma\, \gamma^\nu_{\gamma\delta}\, \psi_\delta\, A_\nu(x) A^\mu(x_3)$$

$$= (-1)^2\, S_{F\alpha\gamma}(x_1 - x)\, ie\, \gamma^\nu_{\gamma\delta}\, S_{F\delta\beta}(x - x_2)\, \Delta_F{}^\mu{}_\nu(x - x_3)$$

Feynman diagram



The vertex gives a factor $ie\gamma^\nu$

Since $S_F(x_1 - x) \neq S_F(x - x_1)$, the fermion propagator has a direction:

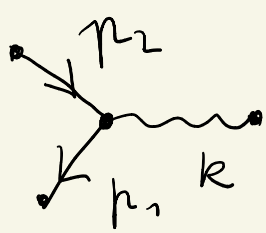
$S_{F_{\alpha\beta}}(x - x') =$ 

The diagram is read against the arrow direction

momentum space:

$$\int d^4x_1 \, d^4x_2 \, d^4x_3 \; e^{i(p_1 x_1 - p_2 x_2 + k \cdot x_3)} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A \; \psi \bar{\psi} A^\mu$$

$$= (2\pi)^4 \delta(p_1 - p_2 + k) S_F(p_1) \, ie\gamma_\nu \, S_F(p_2) \, \Delta^{\mu\nu}(k)$$



LSZ for photons [Srednicki 55]

To obtain the LSZ formula one has to consider operators.

The operator eqs. of motion are Maxwell's eqs.

For $J = 0$:

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu \partial \cdot A = 0$$

Coulomb gauge: $\nabla \cdot \vec{A} = 0$, $\partial . A = \partial_0 A_0$

$\nu = 0$: $\partial^2 A^0 - (\partial^0)^2 A^0 = -\Delta A^0 = 0$ sol: $A^0 = 0$

$\nu = n$: $\partial^2 A^n - \partial^n A^0 = \partial^2 A^n = 0$

general solution (with $k_0 = |\vec{k}|$)

$$\vec{A}(x) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \left\{ e^{-ikx} \vec{\varepsilon}_\lambda(\vec{k}) a_\lambda(\vec{k}) + e^{ikx} \vec{\varepsilon}^{\,*}_\lambda(\vec{k}) a^\dagger_\lambda(\vec{k}) \right\}$$

The Coulomb gauge condition requires $\vec{k} \cdot \vec{\varepsilon}_\lambda(\vec{k}) \Rightarrow$ for each $\vec{k}$ there are two linearly independent polarization vectors. Choose normalization

$$\vec{\varepsilon}_\lambda \cdot \vec{\varepsilon}^{\,*}_{\lambda'} = \delta_{\lambda\lambda'}$$

For $\vec{k} = (0, 0, k)$ one may choose

$$\vec{\varepsilon}_\pm = \frac{1}{\sqrt{2}} (1, \mp i, 0)$$

One can show that then $a^\dagger_\lambda$ creates photons with helicity λ.

For scalars we had $a^{\dagger}(\vec{k}) = i \int d^3x \, \varphi \overleftrightarrow{\partial_t} e^{-ikx}$

Similarly one finds

$$a^{\dagger}_{\lambda}(\vec{k}) = i \, \vec{\varepsilon}_{\lambda}(\vec{k}) \cdot \int d^3x \, \vec{A}(x) \overleftrightarrow{\partial_t} e^{-ikx}$$

From this one can see that the difference compared to scalars is a factor

$\vec{\varepsilon}_{\lambda}(\vec{k})$ in-

     for   state photon w. momentum $\vec{k}$, polarization λ

$\vec{\varepsilon}^{\,*}_{\lambda}(\vec{k})$ out-