6.2 Path integral

path integral $I = \int \mathcal{D}\pi^1 \mathcal{D}\pi^2 \mathcal{D}A^1 \mathcal{D}A^2 \exp\{ i \int dt \, [-H + \int d^3x \, \vec{\pi} \cdot \dot{\vec{A}}] \}$

problem: (i) exponent is not an integral of a local density because E_3 is a non-local functional.

(ii) Lorentz invariance is obscured

Introduce additional integral

$$1 = \int \mathcal{D}\pi_3 \; \delta(\pi_3 + E_3[\vec{\pi}, \rho]) \quad [\pi_3 \text{ is not canonical conjugate of anything}]$$

$$\pi_3 + E_3 = 0 \iff \nabla \cdot \vec{\pi} + \rho = 0$$

$$\delta(\pi_3 + E_3) = \delta(\nabla \cdot \vec{\pi} + \rho) \det\left(\frac{\delta(\nabla \cdot \vec{\pi})}{\delta \pi_3} \right)$$

$$\det\left(\frac{\delta\,\nabla\cdot\vec{\pi}}{\delta\,\pi_3}\right) = \det(\partial_3)$$

The δ-functional can also be written as a path integral

$$\delta(\nabla\cdot\vec{\pi}+\rho) = \int \mathcal{D}A^0 \exp\left\{-i\int d^4x\, A^0(\nabla\cdot\vec{\pi}+\rho)\right\}$$

~~So far~~ Now we have a local action in the exponent:

$$I = \int \mathcal{D}\pi_1\,\mathcal{D}\pi_2\,\mathcal{D}\pi_3\,\mathcal{D}A^0\mathcal{D}A^1\mathcal{D}A^2 \exp\left\{i\int d^4x\left[\vec{\pi}\cdot\dot{\vec{A}} - \tfrac{1}{2}(\vec{\pi}^2+\vec{B}^2) + \vec{J}\cdot\vec{A} - A^0(\nabla\cdot\vec{\pi}+\rho)\right]\right\}\det(\partial_3)$$

Integrate out $\vec{\pi}$:

$$\int d^4x\left[-\tfrac{1}{2}\vec{\pi}^2 + \vec{\pi}\cdot\dot{\vec{A}} + \vec{\pi}\cdot\nabla A_0\right]$$
$$= \int d^4x\left[-\tfrac{1}{2}(\vec{\pi}-\dot{\vec{A}}-\nabla A_0)^2 + \tfrac{1}{2}(\dot{\vec{A}}+\nabla A_0)^2\right]$$

(*) $$I = \int \mathcal{D}A^0\mathcal{D}A^1\mathcal{D}A^2 \exp\left\{i\int d^4x\,\Big[\underbrace{\tfrac{1}{2}\left((-\dot{\vec{A}}-\nabla A_0)^2-\vec{B}^2\right) - J_\mu A^\mu}_{=\mathcal{L}}\Big]\right\}\det(\partial_3)$$

Now introduce integral over A^3:

$$1 = \int \mathcal{D}A^3\,\delta(A^3)$$

Then I is of the form

$$\boxed{I = \int \mathcal{D}A\,\delta(G[A])\,\det\left(\frac{\delta G[A']}{\delta\chi}\right)\exp\left\{i\int d^4x\,\mathcal{L}\right\}}$$

where $A'_\mu = A_\mu \mp \partial_\mu\chi$.

In (*): $G[A] = A_3$

Claim: I is independent of the gauge fixing functional G.

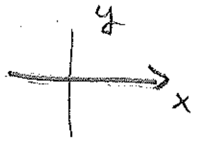
Faddeev-Popov trick. Also works for non-abelian gauge fields, like in QCD.

Check: the G-independence in a 2-dimensional example:

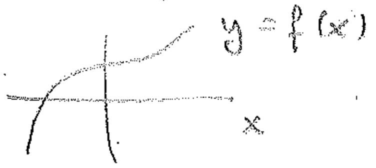
$a = (x, y)$

$I = \int dx\, e^{iS}$ where $S = S(x)$

$I = \int d^2a\, \delta(y)\, e^{iS(x)}$



Now integrate along this arbitrary curve:



$I = \int d^2a\, \delta(y - f(x))\, e^{iS}$

If this curve is uniquely determined by ~~the~~ an implicit relation $G(a) = 0$:

$$I = \int d^2a\, \delta(G(a))\, \frac{\partial G}{\partial y}\, e^{iS}$$, independent of G

We use our freedom to choose

$$G(x) = \phi(x) - \partial_\mu A^\mu(x)$$

with an arbitrary scalar field ϕ. Since this is ϕ-independent, we can integrate over ϕ with the measure

$$\exp\left\{-\frac{1}{2\xi}\int d^4x\, \phi^2\right\}$$

ξ: gauge fixing parameter. $\frac{\delta G}{\delta \chi}$ is an A-independent constant. $\Rightarrow$

$$\boxed{I = \int \mathcal{D}A\, \exp\left\{i\left[S - \frac{1}{2\xi}\int d^4x\, (\partial\cdot A)^2\right]\right\}}$$

Lorentz invariance is now manifest

6.3 Feynman propagator

Scalar field: $S_0 = \frac{1}{2}\int d^4x\, \varphi(-\partial^2 - m^2)\varphi$, $\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$

i.e., the Feynman propagator is i times the inverse of the Fourier transformed differential operator in the quadratic part of the action (without the $\frac{1}{2}$).

For gauge fields with $S_{eff} = S - \frac{1}{2\xi}\int d^4x\, (\partial . A)^2$:

$$S_0 = \int d^4x \left\{ \left(-\tfrac{1}{4}\right)(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \tfrac{1}{2\xi}(\partial . A)^2 \right\}$$

$$= -\frac{1}{2}\int d^4x \left\{ \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{\xi}\partial_\mu A^\mu \partial_\nu A^\nu \right\}$$

$$= \frac{1}{2}\int d^4x\, A^\mu \left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu\right) A^\nu$$

$\Rightarrow$ Feynman propagator

$$\boxed{\Delta_F^{\mu\nu}(p) = \frac{i}{p^2 + i\varepsilon}\left(-\eta^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)}$$

check: $\left(-p^2\eta_{\lambda\mu} + \left(1 - \frac{1}{\xi}\right)p_\lambda p_\mu\right)\frac{1}{i}\Delta_F^{\mu\nu}$

$$= \frac{1}{p^2}\left\{ p^2\delta^\nu_{\ \lambda} - (1-\xi)p_\lambda p^\nu - \left(1 - \frac{1}{\xi}\right)p_\lambda p^\nu + \underbrace{(1-\xi)\left(1-\frac{1}{\xi}\right)}_{= 1 - \frac{1}{\xi} - \xi + 1} p_\lambda p^\nu \right\}$$

$$= \delta^\nu_{\ \lambda}$$

Note that without the gauge fixing term $\frac{1}{\xi}\partial_\mu\partial_\nu$ the differential operator would have a zero-mode and could not be inverted:

$$(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\partial^\nu\chi = 0$$

ξ is a free parameter. $\xi = 0$ <u>Landau gauge</u>

$\xi = 1$ <u>Feynman gauge</u>

physical results like S-matrix elements must be independent of ξ.