5.7 Path integral for Dirac field

[Kaplan, Fermionic path integration]

We found that for fermions in QM = QFT in 0+1 dimensions:

$$\langle \bar\psi_f | e^{-iH(t_f - t_i)} | \psi_i \rangle = \int \mathcal{D}\bar\psi \, \mathcal{D}\psi \, e^{iS}$$

integral over Grassmann-valued paths $\psi(t)$, $\bar\psi(t)$.

boundary conditions: $\psi(t_i) = \psi_i$, $\bar\psi(t_f) = \bar\psi_f$

action: $S = \int dt \, \bar\psi (i\partial_t - m)\psi$

The analogous steps for a Dirac field gives

$$\langle \bar\psi_f | e^{-iH(t_f - t_i)} | \psi_i \rangle = \int \mathcal{D}\bar\psi \, \mathcal{D}\psi \, e^{iS}$$

integral over Grassmann-valued paths $\psi(t, \vec{x})$, $\bar\psi(t, \vec{x})$

boundary conditions: $\psi(t_i, \vec{x}) = \psi_i(\vec{x})$, $\bar\psi(t_f, \vec{x}) = \psi_f(\vec{x})$

action: $S = \int d^4x \, \bar\psi (i\partial\!\!\!/ - m)\psi$

For S-matrix elements we need n-point functions

$$\langle 0 | T\left[\psi(x_1)\cdots\psi(x_k)\bar{\psi}(x_{k+1})\cdots\bar{\psi}(x_n)\right]|0\rangle$$

where for fermionic operators the time ordering is defined as

$$\boxed{T\, A(t)B(t') := \Theta(t-t')\,A(t)\,B(t') - \Theta(t'-t)\,B(t')\,A(t)}$$

Again define a generating functional

$$Z[\eta,\bar{\eta}] := \langle 0 | T \exp\left(i\int d^4x \left[\bar{\eta}\psi + \bar{\psi}\eta\right]\right)|0\rangle$$

with c-number Grassmann field $\eta(x)$, $\bar{\eta}(x)$ such that

$$\langle\langle 0 | T\left[\psi(x_1)\cdots\psi(x_k)\bar{\psi}(x_{k+1})\cdots\bar{\psi}(x_n)\right]|0\rangle$$

$$= \left[\left(-i\frac{\delta}{\delta\eta(x_n)}\right)\cdots\left(-i\frac{\delta}{\delta\bar{\eta}(x_1)}\right) Z[\eta,\bar{\eta}]\right]_{\eta=\bar{\eta}=0}$$

Again by slightly tilting the time contour one obtains path integral for Z:

$$Z[\eta,\bar{\eta}] = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i\int d^4x\left\{\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi - \bar{\psi}\eta\right\}\right)}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i\int d^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi\right)}$$

Complete the square:

$$\int \mathcal{D}\bar\psi \mathcal{D}\psi \exp\left(i \int d^4x \left\{ \bar\psi (i\partial\!\!\!/ - m)\psi + \bar\eta\psi + \bar\psi\eta \right\} \right)$$

$$= \int \mathcal{D}\bar\psi \mathcal{D}\psi \exp\left(i\int d^4x \left\{ \left(\bar\psi + \bar\eta \frac{1}{i\partial\!\!\!/ - m}\right)(i\partial\!\!\!/ - m)\left(\psi + \frac{1}{i\partial\!\!\!/ - m}\eta\right) - \bar\eta \frac{1}{i\partial\!\!\!/ - m}\eta \right\} \right)$$

$$= \exp\left(-i \int d^4x \, \bar\eta \frac{1}{i\partial\!\!\!/ - m} \eta \right) \int \mathcal{D}\bar\psi \mathcal{D}\psi \exp\left(i\int d^4x \, \bar\psi (i\partial\!\!\!/ - m)\psi \right)$$

$\Rightarrow$

$$\boxed{Z[\eta, \bar\eta] = \exp\left(- \int d^4x \, d^4x' \, \bar\eta(x) S_F(x-x') \eta(x') \right)}$$

with the <u>Feynman propagator</u>

$$\boxed{S_F(p) = \frac{i}{p\!\!\!/ - m}}$$

or more precisely

$$\boxed{S_F(p) = i \frac{p\!\!\!/ + m}{p^2 - m^2 + i\varepsilon}}$$

2-point function:

$$\langle 0 | T \, \psi(x_1) \bar\psi(x_2) | 0 \rangle$$

$$= \left(-i \frac{\delta}{\delta\eta(x_2)}\right)\left(-i\frac{\delta}{\delta\bar\eta(x_1)}\right) (-1) \int d^4x \, d^4x' \, \bar\eta(x) S_F(x-x') \eta(x')$$

$$= S_F(x_1 - x_2)$$

LSZ for fermions

S-matrix element for 2 → 2 scattering of scalars:

$$\langle \vec{p}_3 \, \vec{p}_4 \, | S - 1 | \, \vec{p}_1 \, \vec{p}_2 \rangle = (i)^4 \lim_{p_b^0 \to E_{\vec{p}_b}} \prod_{a=1}^{4} (-p_a^2 + m^2)$$

$$\cdot \, G_R^{(4)}(-p_1, -p_2, p_3, p_4)$$

That, is the propagators of the external lines are removed from the n-point function

This is also true for Dirac fermions, but there are additional factors depending on their spin:

		operator in G	factor in S-matrix
in-state	fermion (a)	$\overline{\Psi}$	$u_s(\vec{p})$
out-state	fermion	Ψ	$\overline{u}_s(\vec{p})$
in-state	antifermion (b)	Ψ	$\overline{v}_s(\vec{p})$
out-state	antifermion	$\overline{\Psi}$	$v_s(\vec{p})$

(see e.g. Srednicki 41)

6 Gauge fields

6.1 Canonical formulation

electromagnetic field strength tensor $F^{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) - \rho A^0 + \vec{J}\cdot\vec{A}$$

A^μ: 4-vector potential , $J = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}$ 4-current

A has 4 components, but ∃ only 2 photon polarizations.

$S = \int d^4x\, \mathcal{L}$ invariant under gauge transformations

$$A^\mu \to A^\mu + \partial^\mu \chi$$

if the current is conserved, $\partial_\mu J^\mu = 0$.

gauge transform such that

$$A^3 = 0 \qquad \text{"\underline{axial gauge}"}$$

<u>canonical momenta</u>: $\pi_\mu = \dfrac{\partial \mathcal{L}}{\partial \dot{A}^\mu}$

$\frac{1}{2}\vec{E}^2 = \frac{1}{2}(\partial_i A_0 + \partial_0 A^i)^2$ contains no $\dot{A}^0$ $\Rightarrow$

$\pi_0 = 0$ "<u>constraint</u>"

also: $\pi_3 = 0$

only 2 non-vanishing canonical momenta

$$\pi_i = \partial_0 A^i + \partial_i A_0 \qquad (i = 1, 2)$$

NB: $\pi_i = -E_i \quad (i = 1, 2)$

Gauß' Law: $\nabla\cdot\vec{E} = \rho \Rightarrow -\nabla\cdot\vec{\pi} + \partial_3 E_3 = \rho$ (*)
with $E_3 = -\partial_3 A_0$. This is a constraint for A^0.
It can be solved by x^3-integration. Then A^0 (and E_3) is fixed in terms of $\vec{\Pi}$ and ρ, taken at the same time.

Then we have 2 independent fields A^1, A^2 and 2 canonical momenta π_1, π_2.

$$\mathcal{L} = \tfrac{1}{2}\left[\vec{\pi}^2 + E_3^2 - \vec{B}^2\right] - \rho A^0 + \vec{J}\cdot\vec{A}$$

$$H = \int d^3x \left\{\vec{\pi}\cdot\dot{\vec{A}} - \mathcal{L}\right\} = \int d^3x \left\{\vec{\pi}\cdot(\vec{\pi} - \nabla A_0) - \mathcal{L}\right\}$$

$$-\int d^3x\, \vec{\pi}\cdot\nabla A_0 = \int d^3x\, A_0 \nabla\cdot\vec{\pi} = \int d^3x\, A^0(\partial_3 E_3 - \rho)$$

$$= -\int d^3x\,(E_3\partial_3 A^0 + A^0\rho) = \int d^3x\,(E_3^2 - A^0\rho)$$

$$H = \int d^3x \left\{\tfrac{1}{2}\left[\vec{\pi}^2 + E_3^2 + \vec{B}^2\right] - \vec{J}\cdot\vec{A}\right\}$$