5.5 Quantization, Spin & Statistics

[Coleman 23]
[Peskin, Schroeder 3.5]

canonical quantization of Dirac field:

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi = \psi^+(i[\partial_0 + \gamma^0\vec{\gamma}\cdot\nabla] - \gamma^0 m)\psi$$

Canonical conjugate of ψ: $\quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^+$

Hamiltonian density:

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^+\dot{\psi} - \mathcal{L} = -\overline{\psi}(i\vec{\gamma}\cdot\nabla - m)\psi$$

If ψ solves the EOM $\quad (i\not{\partial} - m)\psi = 0 \quad \Rightarrow$

$$(i\vec{\gamma}\cdot\nabla - m)\psi = -i\gamma^0\dot{\psi}, \quad \text{and}$$

$$\mathcal{H} = -\overline{\psi}(-i\gamma^0\dot{\psi}) = i\psi^+\dot{\psi}$$

With similar arguments as for the scalar field: To obtain something physically sensible, one has no choice whether one uses commutators or anticommutators. Dirac fields must be quantized with anticommutators:

$$\boxed{\begin{aligned}\{\psi_\alpha(t,\vec{x}), \psi^+_\beta(t,\vec{x}')\} &= \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')\\ \{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{x}')\} &= 0\end{aligned}} \qquad (*)$$

The solution of the free Dirac-equation was $(p^0 = \sqrt{\vec{p}^{\,2} + m^2})$

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2p^0}\left\{e^{-ipx} u_s(p)\, a_s(\vec{p}) + e^{ipx} v_s(\vec{p})\, b^+_s(\vec{p})\right\}$$

with operators a, b^+. Then $(*)$ is equivalent to

$$\{a_s(\vec{p}), a^+_{s'}(\vec{p}\,')\} = \{b_s(\vec{p}), b^+_{s'}(\vec{p}\,')\} = 2p^0(2\pi)^3\,\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}$$

all other anticommutators of a, a^+, b, b^+ vanish like for Klein-Gordon field:

$a_s^\dagger(\vec{p})$, $b_s^\dagger(\vec{p})$ create particles with

4-momentum p^μ, $p^0 = \sqrt{\vec{p}^2 + m^2}$

$S = \int d^4x\, \bar{\psi}(i\not{\partial} - m)\psi$ invariant under $\psi \to e^{i\varepsilon}\psi$, $\bar{\psi} \to e^{-i\varepsilon}\bar{\psi}$

$\Rightarrow$ conserved charge Q

a-particles have $Q = 1$
b-particles have $Q = -1$

what is different:

$$|\vec{p}\,s, \vec{p}\,'s'\rangle = \underbrace{a_s^\dagger(\vec{p})\, a_{s'}^\dagger(\vec{p}\,')}_{= -a_{s'}^\dagger(\vec{p}\,')\, a_s^\dagger(\vec{p})} |0\rangle = -|\vec{p}\,'s', \vec{p}\,s\rangle$$

antisymmetric under particle exchange $\Rightarrow$

particles are fermions

In relativistic QM particles with half-integer spin have to be fermions.

"spin-statistic theorem"

5.6 Path integral for fermions

[D. Kaplan] [Srednicki 43-45]

Fermionic harmonic oscillator

$$H = \frac{m}{2}(b^+ b - b b^+) = m\left(b^+ b - \frac{1}{2}\right)$$

$\{b, b\} = 0, \quad \{b, b^+\} = 1$

ground state : $|0\rangle$, $b|0\rangle = 0$

excited state $|1\rangle = b^+|0\rangle$, $b|1\rangle = |0\rangle$

$H|0\rangle = -\frac{m}{2}|0\rangle$, $H|1\rangle = \frac{m}{2}|1\rangle$

The Hilbert space is two-dimensional

define Graßmann numbers $\psi, \bar\psi$ through

$\{\psi, \psi\} = \{\psi, \bar\psi\} = \{\bar\psi, \bar\psi\} = 0$

and that they commute with the operators b, b^+.

$\psi, \bar\psi$ are anticommuting c-numbers.

Define the 'coherent states'

$$|\psi\rangle = e^{-\bar\psi\psi/2}(|0\rangle + \psi|1\rangle), \quad \langle\bar\psi| = e^{-\bar\psi\psi/2}(\langle 0| + \langle 1|\bar\psi)$$

expand the exponentials $\Rightarrow$

$$|\psi\rangle = (1 - \bar\psi\psi/2)|0\rangle + \psi|1\rangle, \quad \langle\bar\psi| = \langle 0|(1 - \bar\psi\psi/2) + \langle 1|\bar\psi$$

$b|\psi\rangle = \psi|0\rangle = \psi|\psi\rangle$ $\qquad |\psi\rangle =$ eigenket of b

$\langle\bar\psi|b^+ = \langle 0|\bar\psi = \langle\bar\psi|\bar\psi$, $\langle\bar\psi| =$ eigenbra of b^+

further properties:

$$(*) \quad \langle \bar\Psi_1 | \psi_2 \rangle = \exp\left\{ -\tfrac{1}{2} \bar\Psi_1 \psi_1 - \tfrac{1}{2} \bar\Psi_2 \psi_2 + \bar\Psi_1 \psi_2 \right\}$$

$$\langle \bar\Psi | \psi \rangle = 1$$

$$|\psi\rangle\langle\bar\Psi| = (1 - \bar\Psi\psi)|0\rangle\langle 0| + \bar\Psi|0\rangle\langle 1| + \psi|1\rangle\langle 0| - \bar\Psi\psi|1\rangle\langle 1|$$

We would like define the integration over Graßmann variables such that the completeness relation

$$\int d\bar\Psi \, d\psi \; |\psi\rangle\langle\bar\Psi| = 1$$

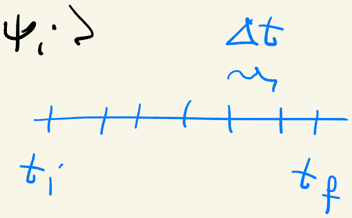
holds. This is achieved if

$$\int d\psi = \partial_\psi$$, that is derivative = integral:

$$\int d\psi = 0, \quad \int d\psi \, \psi = 1 \qquad \left(= |0\rangle\langle 0| + |1\rangle\langle 1| \right)$$

Path integral

We would like to compute $\langle \bar\Psi_f | e^{-iH(t_f - t_i)} | \psi_i \rangle$



$$= \left(e^{-iH\Delta t}\right)^N \quad \text{with} \quad \Delta t = \frac{t_f - t_i}{N}$$

Insert $\int d\bar\Psi_n \, d\psi_n \; |\psi_n\rangle\langle\bar\Psi_n|$ between all factors

Then we obtain (drop the $1/2$ in H)

$$\langle \bar\Psi_n | e^{-iH\Delta t} | \psi_{n-1} \rangle = \langle \bar\Psi_n | e^{-im b^+ b \Delta t} | \psi_{n-1} \rangle$$

$$= \langle \bar\Psi_n | 1 - im\, b^+ b \,\Delta t | \psi_{n-1} \rangle + O(\Delta t^2) = (1 - im \bar\Psi_n \psi_{n-1} \Delta t) \langle \bar\Psi_n | \psi_{n-1} \rangle$$

$$\overset{(*)}{=} \exp\left\{ -im \bar\Psi_n \psi_{n-1} \Delta t - \tfrac{1}{2} \bar\Psi_{n-1} \psi_{n-1} - \tfrac{1}{2} \bar\Psi_n \psi_n + \bar\Psi_n \psi_{n-1} \right\}$$

Re-write the last 3 terms in the exponent.

$$-\frac{1}{2}\left(\bar\psi_{n-1}\psi_{n-1}-\bar\psi_n\psi_{n-1}\right)-\frac{1}{2}\left(\bar\psi_n\psi_n-\bar\psi_n\psi_{n-1}\right)$$

$$=+\frac{1}{2}\left(\bar\psi_n-\bar\psi_{n-1}\right)\psi_{n-1}-\frac{1}{2}\bar\psi_n\left(\psi_n-\psi_{n-1}\right)$$

$$\simeq \Delta t\left[\frac{1}{2}\dot{\bar\psi}\psi-\bar\psi\dot\psi\right]=-\Delta t\,\frac{1}{2}\bar\psi\overleftrightarrow{\partial_t}\psi$$

where $\psi_n=\psi(t_n)$

$$\langle\bar\psi_n|e^{-iH\Delta t}|\psi_{n-1}\rangle\simeq\exp\left\{+i\Delta t\,\bar\psi\left(\frac{i}{2}\overleftrightarrow{\partial_t}-m\right)\psi\right\}$$

and

$$\langle\bar\psi_f|e^{-iH(t_f-t_i)}|\psi_i\rangle=\int\mathcal{D}\bar\psi\,\mathcal{D}\psi\,e^{iS}\quad\text{with}$$

$$S=\int_{t_i}^{t_f}dt\,\bar\psi(t)\left(\frac{i}{2}\overleftrightarrow{\partial_t}-m\right)\psi$$

Boundary conditions: $\bar\psi(t_f)=\bar\psi_f$, $\psi(t_i)=\psi_i$

Choose $\bar\psi(t_f)=\psi(t_i)=0$, integrate by parts $\Rightarrow$

$$S=\int_{t_i}^{t_f}dt\,\bar\psi\left(i\partial_t-m\right)\psi$$

generalization for Dirac field:

$\int \mathcal{D}\bar\psi \, \mathcal{D}\psi \, e^{iS}$ with $S = \int d^4x \, \bar\psi (i\not\partial - m)\psi$

and Graßmann fields $\psi_\alpha(x)$, $\bar\psi_\alpha(x)$.

n-point functions:

$$\boxed{\begin{aligned} &\langle 0 | T \, \psi(x_1) \cdots \psi(x_k) \, \bar\psi(x_{k+1}) \cdots \bar\psi(x_n) | 0 \rangle \\ &= \frac{\int \mathcal{D}\bar\psi \, \mathcal{D}\psi \, e^{iS} \, \psi(x_1) \cdots \bar\psi(x_n)}{\int \mathcal{D}\bar\psi \, \mathcal{D}\psi \, e^{iS}} \end{aligned}}$$

Where for fermionic operators A, B

$$T \, A(t) B(0) = \Theta(t) A(t) B(0) - \Theta(-t) B(0) A(t)$$

Generating functional

$$Z[\eta, \bar\eta] = \frac{\int \mathcal{D}\bar\psi \, \mathcal{D}\psi \exp\left\{ i \int d^4x \left[\bar\psi (i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta \right] \right\}}{\int \mathcal{D}\bar\psi \, \mathcal{D}\psi \exp\left\{ i \int d^4x \, \bar\psi (i\not\partial - m)\psi \right\}}$$

$\langle 0 | T [\psi(x_1) \cdots \bar\psi(x_n)] | 0 \rangle$

$$= \left[\left(-i \frac{\delta}{\delta \bar\eta(x_1)} \right) \cdots \left(+i \frac{\delta}{\delta \eta(x_n)} \right) \right] Z[\eta, \bar\eta] \Big|_{\eta = \bar\eta = 0}$$

note the sign

Completing the square one obtains

$$\boxed{Z[\eta, \bar\eta] = \exp\left\{ -\int d^4x \int d^4x' \, \bar\eta(x) \, S_F(x - x') \, \eta(x') \right\}}$$

with the Feynman propagator

$$S_F(p) = \frac{i}{\not p - m} \quad \text{or, more precisely} \quad S_F(p) = i \frac{\not p + m}{p^2 - m^2 + i\epsilon}$$

5.7 LSZ for fermions

S-matrix element for 2 → 2 scattering of scalars:

$$\langle \vec{p}_3\, \vec{p}_4 \,|\, S-1 \,|\, \vec{p}_1, \vec{p}_2 \rangle = (i)^4 \lim_{p_b^0 \to E_{\vec{p}_b}} \prod_{a=1}^{4} (-p_a^2 + m^2)$$

$$\cdot\, G_R^{(4)}(-p_1, -p_2, p_3, p_4)$$

That, is the propagators of the external lines are removed from the n-point function

This is also true for Dirac fermions, but there are additional factors depending on their spin:

		operator in G	factor in S-matrix
in-state	fermion (a)	$\bar{\psi}$	$u_s(\vec{p})$
out-state	fermion	ψ	$\bar{u}_s(\vec{p})$
in-state	antifermion (b)	ψ	$\bar{v}_s(\vec{p})$
out-state	antifermion	$\bar{\psi}$	$v_s(\vec{p})$

(see e.g. Srednicki 41)