5.3 Scalar and vectors made of spinors

Hermitean conjugate γ-matrices:

$$\boxed{(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0}$$

check: $(\gamma^0)^+ = \gamma^0 = \gamma^0 \underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}$, $(\gamma^m)^+ = -\gamma^m = -\overbrace{\gamma^m\gamma^0}^{=-\gamma^0\gamma^m}\gamma^0 = \gamma^0\gamma^m\gamma^0$ □

$(S^{\mu\nu})^+ = -\frac{i}{4}[\gamma^\mu,\gamma^\nu]^+ = +\frac{i}{4}[\gamma^{\mu +},\gamma^{\nu +}] = \frac{i}{4}\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0$
$= \gamma^0 S^{\mu\nu}\gamma^0$

consequence for $D = \exp\{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\}$:

$D^+ = \exp(\frac{i}{2}\omega_{\mu\nu}\underbrace{(S^{\mu\nu})^+}_{=\gamma^0 S^{\mu\nu}\gamma^0}) = \gamma^0 \exp(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu})\gamma^0 \implies$

$$\boxed{D^+ = \gamma^0 D^{-1}\gamma^0}$$

def. $\bar\psi := \psi^+\gamma^0$

LT: $\bar\psi \to \psi^+ D^+\gamma^0 = \psi^+\gamma^0\gamma^0 D^+\gamma^0 = \bar\psi D^{-1}$, $\psi \to D\psi \implies$

$$\boxed{\bar\psi\psi \text{ is a scalar}}$$

furthermore:

$$\boxed{\bar\psi\gamma^\rho\psi \text{ is a 4-vector}}$$

check: $\bar\psi\gamma^\rho\psi \to \bar\psi D^{-1}\gamma^\rho D\psi$

for infinitesimal ω:

$D^{-1}\gamma^\rho D = (1+\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu})\gamma^\rho(1-\frac{i}{2}\omega_{\lambda\tau}S^{\lambda\tau})$
$= \gamma^\rho + \frac{i}{2}\omega_{\mu\nu}[S^{\mu\nu},\gamma^\rho]$

problem 10.1: $[S^{\mu\nu},\gamma^\rho] = -i(\eta^{\mu\rho}\gamma^\nu - (\mu\leftrightarrow\nu)) \implies$

$D^{-1}\gamma^\rho D = \gamma^\rho + \frac{1}{2}\omega_{\mu\nu}(\eta^{\mu\rho}\gamma^\nu - (\mu\leftrightarrow\nu))$
$= \gamma^\rho + \omega^\rho{}_\nu\gamma^\nu = \Lambda^\rho{}_\nu\gamma^\nu$ □

5.4 Free Dirac-equation

[Peskin 3.2, 3.3]
[Wachtmann 4.2]

consider a Dirac-spinor field $\psi(x)$

we want to find a Lorentz-invariant action S.
This is achieved when $S = \int d^4x\, \mathcal{L}$ with Lorentz invariant $\mathcal{L}$. $\bar{\psi}\psi$ is a scalar.
We also need derivatives to obtain a non-trivial eq. of motion.
∂_μ transforms like a 4-vector with lower index,
$\partial_\mu \to (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu$
We have seen that $\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu{}_\nu\, \bar{\psi}\gamma^\nu\psi \implies$

$\bar{\psi}\gamma^\mu\partial_\mu\psi$ is a scalar.

Our two scalars can be combined to

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

define $\not{\partial} := \gamma^\mu\partial_\mu$ "d-slash", so that

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi}$$

Then S is Hermitean:

$$S^\dagger = \int d^4x \Big\{ -i(\partial_\mu\psi^\dagger)\underbrace{\gamma^{\mu\dagger}}_{=\gamma^0\gamma^\mu\gamma^0}\underbrace{\bar{\psi}^\dagger}_{=(\psi^\dagger\gamma^0)^\dagger = \gamma^0\psi} - m\psi^\dagger\bar{\psi}^\dagger \Big\}$$

$$= \int d^4x \left\{ -i(\partial_\mu\bar{\psi})\psi - m\bar{\psi}\psi \right\} = S$$

integrate by parts

eq. of motion: $\delta S = 0$, treat $\psi, \bar\psi$ as independent

(✻) $\Rightarrow$ $\boxed{(i\not\partial - m)\psi = 0}$ free Dirac-equation

Solving the free Dirac eq

multiply by $(-i\not\partial - m) \Rightarrow$ $(\not\partial^2 - m^2)\psi = 0$

For any a^μ: $\not a^2 = a_\mu a_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} a_\mu a_\nu \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{= 2\eta^{\mu\nu}} \Rightarrow$

$\not a^2 = a^2 \mathbb{1}$ $\Rightarrow$

(✱) $(\partial^2 + m^2)\psi = 0$

That is, a solution to the Dirac equation is a solution to the Klein-Gordon equation.

plane wave Ansatz: $\psi(x) = e^{-ipx} u(p)$, $p^0 > 0$

"positive energy-solution"

(✱) $\Rightarrow$ $p^0 = \sqrt{\vec p^{\,2} + m^2}$

Insert this in (✻) $\Rightarrow$

$(\not p - m) u(p) = 0$

First consider rest frame $\vec p = 0$.

$(m\gamma^0 - m) u = m(\gamma^0 - \mathbb{1}) u = 0$

Weyl rep.: $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u = 0$$

Solution: $u = u(\vec{0}) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ for any 2-component spinor.

choose normalization: $\xi^\dagger \xi = 1 \Rightarrow$

$$\bar{u}(\vec{0})\, u(\vec{0}) = m(\xi^\dagger \xi + \xi^\dagger \xi) = 2m$$

We have 2 linearly independent solutions. We will see that they correspond to the two orientations of spin 1/2.

For general $\vec{p}$ the solutions can be obtained by a boost

$$u(\vec{p}) = \mathbb{D}\, u(\vec{0})$$

where now $\mathbb{D} = D(\Lambda)$

with $p = \Lambda k$, $k = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$

Lorentz-invariant normalization:

$$\boxed{\bar{u}(\vec{p})\, u(\vec{p}) = 2m}$$

negative energy solutions: $\psi(x) = e^{ipx} v \quad (p^0 > 0)$

$$(\not{p} + m)\, v = 0$$

rest frame: $(m\gamma^0 + m) v = 0$, $\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} v = 0$

$v = v(\vec{0}) = \sqrt{m} \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}$ with any 2-component spinor λ.

choose $\lambda^+ \lambda = 1$

$$\bar{v}(\vec{0})\, v(\vec{0}) = m\, (\lambda^+, -\lambda^+) \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}}_{= \begin{pmatrix} -\lambda \\ \lambda \end{pmatrix}} = -2m$$

Again, for any $\vec{p}$, one obtains $v(\vec{p})$ by a Lorentz boost.

Normalization

$$\boxed{\bar{v}(\vec{p})\, v(\vec{p}) = -2m}$$

Furthermore

$$\bar{u}(\vec{p})\, v(\vec{p}) = \bar{v}(\vec{p})\, u(\vec{p}) = 0$$

For the general solution of the Dirac eq. we need two linearly independent ξ_s, λ_s $(s=1,2)$ which we choose such that $\xi_s^+ \xi_{s'} = \lambda_s^+ \lambda_{s'} = \delta_{ss'}$ $\Rightarrow$

$$\sum_{s=1}^{2} \xi_s \xi_s^+ = \sum_{s=1}^{2} \lambda_s \lambda_s^+ = \mathbb{1}$$

(completeness relations)

The general solution of the free Dirac-equation is a superposition of all positive and negative energy plane waves,

$$\boxed{\psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3 2p^0} \left\{ e^{-ipx} u_s(\vec{p})\, a_s(\vec{p}) + e^{ipx} v_s(\vec{p})\, b_s^*(\vec{p}) \right\}}$$

with Fourier coefficients $a_s(\vec{p})$ and $b_s(\vec{p})$, and with $p^0 = \sqrt{\vec{p}^2 + m^2}$.

We will encounter the spin sums

$$\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = D \sum_s u_s(\vec{0})\bar{u}_s(0) D^{-1}$$

$$\sum_s u_s(\vec{0})\bar{u}_s(\vec{0}) = m\sum_s \begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix}(\xi_s^+, \xi_s^+)\gamma^0$$

$$= m\sum_s \begin{pmatrix}\xi_s\xi_s^+ & \xi_s\xi_s^+\\ \xi_s\xi_s^+ & \xi_s\xi_s^+\end{pmatrix} = m\begin{pmatrix}\mathbb{1} & \mathbb{1}\\ \mathbb{1} & \mathbb{1}\end{pmatrix} = m(\gamma^0+1)$$

$$= (\not{k}+m)$$

We had $\quad D^{-1}\gamma^\mu D = \Lambda^\mu{}_\nu\gamma^\nu \Rightarrow D^{-1}\gamma_\mu D = (\Lambda^{-1})^\nu{}_\mu\gamma_\nu$

$$D\not{k}D^{-1} = k^\mu D\gamma_\mu D^{-1} = k^\mu\Lambda^\nu{}_\mu\gamma_\nu$$

$$= p^\nu\gamma_\nu = \not{p} \qquad \Rightarrow$$

$$\boxed{\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = \not{p}+m}$$

similarly for v:

$$\sum_s v_s(\vec{0})\bar{v}_s(\vec{0}) = m\sum_s \begin{pmatrix}\eta_s\\ -\eta_s\end{pmatrix}(\eta_s^+, -\eta_s^+)\gamma^0$$

$$= m\sum_s \begin{pmatrix}\eta_s\\ -\eta_s\end{pmatrix}(-\eta_s^+, \eta_s) = m\sum_s \begin{pmatrix}-\eta_s\eta_s^+ & \eta_s\eta_s^+\\ \eta_s\eta_s^+ & -\eta_s\eta_s^+\end{pmatrix}$$

$$= m\begin{pmatrix}-\mathbb{1} & \mathbb{1}\\ \mathbb{1} & -\mathbb{1}\end{pmatrix} = m(\gamma^0-1) = \not{k}-m \qquad \Rightarrow$$

$$\boxed{\sum_{s=1}^{2} v_s(\vec{p})\bar{v}_s(\vec{p}) = \not{p}-m}$$