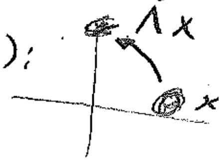
5 Spin ½ fields and particles

[Weinberg 2.4, 2.7]
[Peskin, Schroeder 3.1, 3.2]

We already know different types of fields transforming differently under Lorentz transformations (LT):



$$\varphi(x) \mapsto \varphi(\Lambda^{-1}x) \qquad \text{scalar field}$$

$$A^\mu(x) \mapsto \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}x) \qquad \text{(four-) vector field}$$

Furthermore we has seen that quantization of scalar fields gives scalar (= spin 0) particles.

Both φ and A transform linearly under LT, like

$$\phi(x) \mapsto D(\Lambda)\, \phi(\Lambda^{-1}x) \quad , \text{ where the}$$

matrices D are linear <u>representations</u> of the Lorentz group, i.e. they satisfy

$$(*) \quad D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2), \quad D(\Lambda^{-1}) = (D(\Lambda))^{-1}, \quad D(1) = \mathbb{1}$$

Expectation: quantizing spin-½ fields gives spin-½ particles.

5.1 Spinors

Before getting to the Lorentz group, consider again the rotation group $SO(3)$.

A representation of (*) is given by $\frac{\sigma^i}{2}$ with the Pauli matrices σ^i:

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\,\varepsilon^{ijk}\,\frac{\sigma^k}{2}$$

Exponentiation gives the so-called Spin $\frac{1}{2}$ representation of $SO(3)$:

$$D(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2}$$

The 2-component objects transforming under this rep.

$$\psi \to D(\vec{\theta})\,\psi$$

are called 2-spinors.

example: wave function of a spin $1/2$ particle in non-relativistic QM

This is not a representation in the strict sense:

a rototation by 2π around the x^3 axis is represented by

$$\exp\{i\, 2\pi \tfrac{\sigma^3}{2}\} = \exp\{i\pi\sigma^3\} = -\mathbb{1}$$

It is called a <u>representation up to a phase</u> or <u>projective representation</u>.

Generalization to the Lorentz group:

6 generators $J^{\mu\nu}$. Lie algebra (see problem sheet 4)

$$(*) \quad \boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left[(\eta^{\nu\rho} J^{\mu\sigma} - (\mu\leftrightarrow\nu)) - (\rho\leftrightarrow\sigma)\right]}$$

If one has matrices γ^μ which satisfy

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}} \quad \underline{\text{Clifford algebra}}$$

then

$$\boxed{S^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

is a representation of $(*)$.

proof: problem sheet 11

Usually one writes

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$$

One can show that the reps. of the Clifford algebra are at least 4-dimensional.

explicit representations:

(i) $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$ with $\sigma^0 = \bar\sigma^0 = \mathbb{1}$, $\bar\sigma^m = -\sigma^m$

<u>Weyl rep.</u>

(ii) $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $\qquad \gamma^m = \begin{pmatrix} 0 & \sigma^m \\ -\sigma^m & 0 \end{pmatrix}$

<u>Dirac rep.</u>

using $\{\sigma^m, \sigma^n\} = 2\delta^{mn}\mathbb{1}$ one can easily check that (i) and (ii) satisfy (*).

$$\boxed{D(\omega) = \exp\{-\tfrac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\}}$$

4-component objects ψ transforming like $\psi \to D\psi$ are called <u>Dirac spinors</u>.

in the Weyl rep:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix} - (\mu \leftrightarrow \nu)$$

$$= \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$

is block-diagonal. If we write

$\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$, then the 2-component spinors χ, η transform independently under $SO(1,3)$

[our representation of $SO(1,3)$ is <u>reducible</u>]

χ, η are called <u>Weyl spinor</u>.

under rotations ($\omega_{0\nu} = 0$) χ and η transform in the same way, like 3-spinors.